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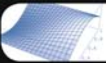
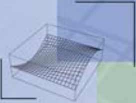


Volume I ANALYTIC METHODS

C. Constanda, M.E. Pérez, EDITORS



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Integral Methods in Science and Engineering Volume 1

Analytic Methods

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Preface

The international conferences on Integral Methods in Science and Engineering (IMSE) are biennial opportunities for academics and other researchers whose work makes essential use of analytic or numerical integration methods to discuss their latest results and exchange views on the development of novel techniques of this type.

The first two conferences in the series, IMSE1985 and IMSE1990, were hosted by the University of Texas–Arlington. At the latter, the IMSE consortium was created and charged with organizing these conferences under the guidance of an International Steering Committee. Subsequently, IMSE1993 took place at Tohoku University, Sendai, Japan, IMSE1996 at the University of Oulu, Finland, IMSE1998 at Michigan Technological University, Houghton, MI, USA, IMSE2000 in Banff, AB, Canada, IMSE2002 at the University of Saint-Étienne, France, IMSE2004 at the University of Central Florida, Orlando, FL, USA, and IMSE2006 at Niagara Falls, ON, Canada. The IMSE conferences are now recognized as an important forum where scientists and engineers working with integral methods express their views about, and interact to extend the practical applicability of, a very elegant and powerful class of mathematical procedures.

A distinguishing characteristic of all the IMSE meetings is their general atmosphere—a blend of utmost professionalism and a strong collegial-social component. IMSE2008, organized at the University of Cantabria, Spain, and attended by delegates from twenty-seven countries on five continents, maintained this tradition, marking another unqualified success in the history of the IMSE consortium. For the smoothness and detail-perfect arrangements throughout the conference, the participants and the Steering Committee would like to express their special thanks to the Local Organizing Committee:

M. Eugenia Pérez (Departamento de Matemática Aplicada y Ciencias de la Computación, ETSI Caminos, Canales y Puertos), *Chairman;*

Miguel Lobo (Departamento de Matemáticas, Estadística y Computación, Facultad de Ciencias);

Delfina Gómez (Departamento de Matemáticas, Estadística y Computación, Facultad de Ciencias).

The Local Organizing Committee and the Steering Committee also wish to acknowledge the financial support received from the following institutions:

Universidad de Cantabria (in particular, Vicerrectorado de Investigación y Transferencia del Conocimiento, Facultad de Ciencias, ETSI Caminos, Canales y Puertos, Departamento de Matemáticas, Estadística y Computación, and Departamento de Matemática Aplicada y Ciencias de la Computación);

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Sociedad Española de Matemática Aplicada (SeMA).

Last but not least, they would like to express their thanks to MICINN (MTM2005-07720) for partial support, to Antonio José González for his work on the graphical design of the conference, to the colleagues—especially Doina Cioranescu—involved in the coordination of the monographic sessions, and to all the participants, whose presence and scientific activity in Santander ensured the success of this meeting.

The next IMSE conference will be held in July 2010 in Brighton, UK. Details concerning this event are posted on the conference web page,

http://www.cmis.brighton.ac.uk/imse2010

This volume contains four invited papers and twenty-seven contributed peer-reviewed papers, arranged in alphabetical order by (first) author's name. The editors would like to thank the staff at Birkhäuser Boston for their efficient handling of the publication process.

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1

Homogenization of the Integro-Differential Burgers Equation

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1.1 Introduction

The Burgers equation is a fundamental partial differential equation of fluid mechanics and acoustics. It occurs in various areas of applied mathematics, such as the modeling of gas dynamics and traffic flow (see [Ho50] and [Co51]).

We consider the integro-differential Burgers equation

$$\frac{\partial u}{\partial x} - \beta \frac{\partial^2 u}{\partial y^2} + \alpha \frac{\partial}{\partial y} f(u) = \nu \frac{\partial}{\partial y} \int_{-\infty}^{y} \frac{\partial u}{\partial y}(x, y') e^{(y'-y)/\tau}\, dy'. \tag{1.1}$$

Function f in the classical setting has the quadratic shape: $f(u) = 0.5u^2$; the integral term on the right-hand side describes the relaxation (memory) effects. The equation is derived by Rudenko and Soluyan from the state equation and the motion equation for a medium with relaxation [RuSo75], see also [PoSo62]. In [La97] Chapter 5, Section 7 this equation is called the Witham–Rudenko equation.

Here $0 < \tau$ is a constant. Equation (1.1) is set in the domain $Q = Q_X = \mathbb{R} \times (0, X)$ and it is supplied with the initial condition for $x = 0$:

$$u(0, y) = \varphi(y), \quad y \in \mathbb{R} \tag{1.2}$$

and the periodicity condition in variable y:

$$u(x, y + 1) = u(x, y), \quad (x, y) \in Q. \tag{1.3}$$

Let us mention that the physical sense of variables x and y is quite opposite to their mathematical sense, i.e., y is the time (more exactly the sound beam time), and x stands for the vertical axis variable. Condition (1.3) corresponds to the periodic regime in time. In the case when the medium is stratified, the coefficients of the equation oscillate. If the scale of variation of properties is

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much less than the macroscopic scale that is normally the height of the sound source, then their ratio is a small dimensionless parameter δ, and the equation takes the form

$$\frac{\partial u}{\partial x}-\beta\left(\frac{x}{\delta}\right)\frac{\partial^2 u}{\partial y^2}+\alpha\left(\frac{x}{\delta}\right)\frac{\partial}{\partial y}f(u)=\nu\left(\frac{x}{\delta}\right)\frac{\partial}{\partial y}\int\limits_{-\infty}^{y}\frac{\partial u}{\partial y}(x,y')e^{(y'-y)/\tau}\,dy', \quad (1.4)$$

where $0<\delta$ stands for the small parameter, the ratio of scales.

As far as we know, the mathematical analysis of problem (1.4), (1.2), (1.3) was first developed in [PaPs08], although the equation there was not really Burgers because there was a Lipschitz condition on function f:

$$|f(u_1)-f(u_2)|\le L|u_1-u_2| \quad \forall u_1,u_2\in\mathbb{R},$$

and so, it could not have a quadratic shape. That is why equation (1.4) was called there "the Burgers-type equation." Apart from this, there were the assumptions that f is a three times continuously differentiable function and that the initial data $\varphi\in H^3_{per}(\mathbb{R})$.

In [PaPs08] an asymptotic approximation for the exact solution u of problem (1.4), (1.2), (1.3) was sought in the form

$$u_a(x,y)=u_0(x,y)+\delta\, u_1(x,y,\xi)|_{\xi=x/\delta}. \quad (1.5)$$

Here u_0 is a solution of the homogenized problem

$$\frac{\partial u_0}{\partial x}-\langle\beta\rangle\frac{\partial^2 u_0}{\partial y^2}+\langle\alpha\rangle\frac{\partial}{\partial y}f(u_0)=\langle\nu\rangle\frac{\partial}{\partial y}\int\limits_{-\infty}^{y}\frac{\partial u_0}{\partial y}(x,y')e^{(y'-y)/\tau}\,dy', \quad (1.6)$$

$$u_0(0,y)=\varphi(y), \quad y\in\mathbb{R}, \quad (1.7)$$

$$u_0(x,y+1)=u_0(x,y), \quad (x,y)\in Q \quad (1.8)$$

with constant coefficients

$$\langle\alpha\rangle=\lim_{T\to\infty}\frac{1}{T}\int_0^T\alpha(\xi)\,d\xi, \langle\beta\rangle=\lim_{T\to\infty}\frac{1}{T}\int_0^T\beta(\xi)\,d\xi, \langle\nu\rangle=\lim_{T\to\infty}\frac{1}{T}\int_0^T\nu(\xi)\,d\xi \quad (1.9)$$

and u_1 is defined by the following formula:

$$u_1(x,y,\xi)=\widetilde{\beta}(\xi)\frac{\partial^2 u_0}{\partial y^2}(x,y)-\widetilde{\alpha}(\xi)\frac{\partial}{\partial y}f(u_0(x,y)) + \widetilde{\nu}(\xi)\frac{\partial}{\partial y}\int\limits_{-\infty}^{y}\frac{\partial u_0}{\partial y}(x,y')e^{(y'-y)/\tau}\,dy', \quad (1.10)$$

where

$$\widetilde{\beta}(\xi)=\int_0^\xi[\beta(t)-\langle\beta\rangle]\,dt, \quad \widetilde{\alpha}(\xi)=\int_0^\xi[\alpha(t)-\langle\alpha\rangle]\,dt. \quad \widetilde{\nu}(\xi)=\int_0^\xi[\nu(t)-\langle\nu\rangle]\,dt.$$

The following estimate was proved in [PaPs08] for the difference between the exact solution and the asymptotic solution in the energy norm:

$$\|u - u_a\|_{V_{per}(Q)} \leq C\delta^{1-d}. \tag{1.11}$$

The goal of this chapter is the analysis of the existence and uniqueness of problem (1.1)–(1.3) in a general setting, when $f \in C^1(\mathbb{R})$, $\varphi \in H^1_{per}(\mathbb{R})$ without any assumptions on the Lipschitz property of f. The central point in this generalization is the proof of the maximum principle for problem (1.1)–(1.3) (Theorem 1).

Moreover, we prove the estimate of error in the $L^2_{per}(Q)$-norm:

$$\|u - u_0\|_{L^2_{per}(Q)} \leq C\delta^{1-d}. \tag{1.12}$$

If $f \in C^2(\mathbb{R})$, $\varphi \in H^2_{per}(\mathbb{R})$, then we prove the estimate

$$\|u - u_0\|_{V_{per}(Q)} \leq C\delta^{1-d} \tag{1.13}$$

of the same order with respect to δ, as in (1.11), (1.12).

We emphasize here that the asymptotic approximation in estimates (1.12), (1.13) is the solution u_0 of the homogenized problem, and not approximation (1.5) containing the term (1.10).

A detailed statement of these results will be published in [AmPa09].

1.2 Notation

In what follows all derivatives are understood as weak derivatives, and we use the following notation:

$$v_x = \frac{\partial v}{\partial x}, \quad v_y = \frac{\partial v}{\partial y}, \quad v_{yy} = \frac{\partial^2 v}{\partial y^2}, \quad v_{xy} = \frac{\partial^2 v}{\partial y \partial x}, \quad v_{yyy} = \frac{\partial^3 v}{\partial y^3}.$$

Let us introduce the functional spaces used in the chapter.

Let $C_{per}(\mathbb{R})$ be the space of continuous on $\mathbb{R}$ periodic with period equal 1 functions. Denote

$$\|v\|_{C_{per}(\mathbb{R})} = \max_{y \in [0,1]} |v(y)|.$$

Let $L^2_{per}(\mathbb{R})$ be the space of measurable on $\mathbb{R}$ periodic with period 1 functions v having the finite norm

$$\|v\|_{L^2_{per}(\mathbb{R})} = \|v\|_{L^2(0,1)}.$$

Denote

$$(u, v) = \int_0^1 u(y)v(y)\, dy.$$

Let $H^m_{per}(\mathbb{R})$ be the space of functions $u \in L^2_{per}(\mathbb{R})$ such that there exist the derivatives $\dfrac{d^k u}{dy^k} \in L^2_{per}$ for all $k = 1, \dots, m$.

Introduce the space $C_{per}(\overline{Q})$ of continuous on $\overline{Q}$ functions $v(x,y)$, periodic in y with the period equal to 1. Define

$$\|v\|_{C_{per}(\overline{Q})} = \max_{(x,y)\in[0,X]\times[0,1]} |v(x,y)|.$$

Introduce the spaces

$$L^{p,2}_{per}(Q) = L^p(0,X;L^2_{per}(\mathbb{R})), \quad L^2_{per}(Q) = L^{2,2}_{per}(Q),$$
$$V_{per}(Q) = C([0,X];L^2_{per}(\mathbb{R})) \cap L^2(0,X;H^1_{per}(\mathbb{R}))$$

with the norms

$$\|u\|_{L^{p,2}_{per}(Q)} = \big\| \|u\|_{L^2_{per}(\mathbb{R})} \big\|_{L^p(0,X)}, \quad \|u\|_{L^2_{per}(Q)} = \|u\|_{L^{2,2}_{per}(Q)},$$
$$\|u\|_{V_{per}(Q)} = \max_{x\in[0,X]} \|u(x,\cdot)\|_{L^2_{per}(\mathbb{R})} + \|u_y\|_{L^2_{per}(Q)}.$$

Denote

$$(u,v)_Q = \int_Q u(x,y)v(x,y)\,dxdy.$$

Let $H^{1,2}_{per}(Q)$ be the space of functions $u \in L^2_{per}(Q)$ such that there exist the derivatives $u_x, u_{yy} \in L^2_{per}(Q)$.

Define the integral operators J and J^* on $L^2_{per}(\mathbb{R})$:

$$J[v](y) = \int_{-\infty}^{y} v(y')e^{(y'-y)/\tau}\,dy', \quad J^*[v](y) = \int_{y}^{\infty} v(y')e^{(y-y')/\tau}\,dy'.$$

Operator J^* is the adjoint operator for J.

Note that using this notation we may rewrite equation (1.1) in the form

$$u_x - \beta u_{yy} + \alpha f(u)_y = \nu J[u_y]_y.$$

1.3 The Integro-Differential Burgers Equation:Existence, Uniqueness, and Smoothness of Solutions

Assume that the following conditions hold:

$$\varphi \in H^1_{per}(\mathbb{R}), \quad \|\varphi\|_{C_{per}(\mathbb{R})} \le N, \quad f \in C^1(\mathbb{R}), \tag{1.14}$$
$$\alpha, \nu \in L^2(0,X), \quad \beta \in L^\infty(0,X), \tag{1.15}$$
$$0 < \kappa_1 \le \beta(x) \le \kappa_2, \ \|\alpha\|_{L^2(0,X)} \le \kappa_2, \ 0 \le \nu(x), \ \|\nu\|_{L^2(0,X)} \le \kappa_2. \tag{1.16}$$

Here κ_1, κ_2, N are some constants.

Let $C = C(\kappa_1, \kappa_2, N)$ be the notation for non-decaying functions of parameters $\kappa_1^{-1}, \kappa_2, N$. If these functions depend as well on function f or on function f and on value X, then we will use the notation $C_f = C_f(\kappa_1, \kappa_2, N)$ or $C_{f,X} = C_{f,X}(\kappa_1, \kappa_2, N)$, respectively. Arguments κ_1, κ_2, N will usually be omitted.

The following version of the maximum principle holds for problem (1.1)–(1.3).

Theorem 1 *Assuming conditions (1.14)–(1.16) consider* $u \in H^{1,2}_{per}(Q)$ *the solution of problem (1.1)–(1.3). The following estimate holds:*

$$\|u\|_{C_{per}(\overline{Q})} \le \|\varphi\|_{C_{per}(\mathbb{R})}. \tag{1.17}$$

Using Theorem 1 and the Galerkin method, we prove the following result about the existence, uniqueness, and additional smoothness of the solution of problem (1.1)–(1.3).

Theorem 2 *Assume that conditions (1.14)–(1.16) hold. Then solution* $u \in H^{1,2}_{per}(Q)$ *of problem (1.1)–(1.3) exists, is unique, and satisfies estimate (1.17) and estimate*

$$\|u_x\|_{L^2_{per}(Q)} + \|u_{yy}\|_{L^2_{per}(Q)} + \|u_y\|_{L^{\infty,2}_{per}(Q)} \le C_f \|\varphi_y\|_{L^2_{per}(\mathbb{R})}.$$

If, in addition,

$$\varphi \in H^2_{per}(\mathbb{R}), \quad \|\varphi_y\|_{L^2_{per}(\mathbb{R})} \le N, \quad f \in C^2(\mathbb{R}), \tag{1.18}$$

then $u_y \in H^{1,2}_{per}(Q)$ *and the following estimate holds:*

$$\|u_{xy}\|_{L^2_{per}(Q)} + \|u_{yyy}\|_{L^2_{per}(Q)} + \|u_{yy}\|_{L^{\infty,2}_{per}(Q)} \le C_f \|\varphi_{yy}\|_{L^2_{per}(\mathbb{R})}.$$

1.4 Stability of the Solution of Problem (1.1)–(1.3)

Let us formulate two results on the stability of the solution of problem (1.1)–(1.3) with respect to the discrepancy. We need these results for the derivation of the error estimate for an asymptotic approximation.

Let

$$f_N(u) = \begin{cases} f(u), & -N \le u \le N, \\ f(-N) + f'(-N)(u+N), & u < -N, \\ f(N) + f'(N)(u-N), & u > N. \end{cases}$$

Theorem 3 *Assume that conditions (1.14)–(1.16) hold and that there exists* $r \in (2, \infty]$ *such that*

$$\alpha \in L^r(0, X), \quad \|\alpha\|_{L^r(0,X)} \leq \kappa_2.$$

Let $u \in H^{1,2}_{per}(Q)$ *be a solution of problem (1.1)–(1.3), and let function* $v \in L^2_{per}(Q)$ *satisfy for all* $t \in (0, X]$ *the following integral identity:*

$$\begin{aligned}&-(v, \psi_x + \beta\psi_{yy} + \nu J^*[\psi_y]_y)_{Q_t} - (\alpha f_N(v), \psi_y)_{Q_t} = \\ &(\varphi, \psi|_{x=0}) + (g^a, \psi_y)_{Q_t} - (g^b, \psi_{yy})_{Q_t} - (g^c, J^*[\psi_y]_y)_{Q_t} \\ &\qquad\qquad \forall \psi \in H^{1,2}_{per}(Q_t), \quad \psi|_{x=t} = 0,\end{aligned} \tag{1.19}$$

where $g^a \in L^{1,2}_{per}(Q)$, $g^b \in L^2_{per}(Q)$, $g^c \in L^{1,2}_{per}(Q)$.

Then the following estimate holds:

$$\|v - u\|_Q \leq C_{f,X,r}\left(\|g^a\|_{L^{1,2}_{per}(Q)} + \|g^b\|_{L^2_{per}(Q)} + \|g^c\|_{L^{1,2}_{per}(Q)}\right). \tag{1.20}$$

Theorem 4 *Assume that conditions (1.14)–(1.16) hold. Let* $u \in H^{1,2}_{per}(Q)$ *be a solution of problem (1.1)–(1.3), and function* $v \in L^2(0, T; H^1_{per}(\mathbb{R}))$ *for* $t = X$ *be a solution of integral identity (1.19).*

Then $u - v \in V_{per}(Q)$ *and the following estimate holds:*

$$\|v - u\|_{V_{per}(Q)} \leq C_f\left(\|g^a\|_{L^2_{per}(Q)} + \|g^b_y\|_{L^2_{per}(Q)} + \|g^c\|_{L^2_{per}(Q)}\right). \tag{1.21}$$

1.5 Problem with Rapidly Oscillating Coefficients

Let $\beta_\delta(x) = \beta(x/\delta)$, $\alpha_\delta(x) = \alpha(x/\delta)$, $\nu_\delta(x) = \nu(x/\delta)$, where $\delta > 0$ is a small parameter.

Assume that the following conditions hold (here $\mathbb{R}^+ = (0, +\infty)$):

$$\beta \in L^\infty(\mathbb{R}^+), \quad \alpha, \nu \in L^2_{loc}(\mathbb{R}^+), \tag{1.22}$$

$$0 < \kappa_1 \leq \beta(\xi) \leq \kappa_2, \ \|\alpha\|_{L^2(0,\xi)} \leq \kappa_2 \xi^{1/2}, \quad \forall \xi \in \mathbb{R}^+, \tag{1.23}$$

$$0 \leq \nu(\xi), \ \|\nu\|_{L^2(0,\xi)} \leq \kappa_2 \xi^{1/2} \ \ \forall \xi \in \mathbb{R}^+. \tag{1.24}$$

It follows from (1.22)–(1.24) that $\beta_\delta \in L^\infty(0, X)$, $\alpha_\delta, \nu_\delta \in L^2(0, X)$, and

$$0 < \kappa_1 \leq \beta_\delta(x) \leq \kappa_2, \ \|\alpha_\delta\|_{L^2(0,X)} \leq \kappa_2 X^{1/2}, \ 0 \leq \nu_\delta(x), \ \|\nu_\delta\|_{L^2(0,X)} \leq k_2 X^{1/2}.$$

So the results of Section 1.3 imply the following theorem.

Theorem 5 *Assume that conditions (1.14), (1.22)–(1.24) hold. Then there exists a unique solution* $u \in H^{1,2}_{per}(Q)$ *of problem (1.4), (1.2), (1.3), and it satisfies the following estimates uniform with respect to* δ*:*

$$\|u\|_{C_{per}(\overline{Q})} \leq \|\varphi\|_{C_{per}(\mathbb{R})},$$

$$\|u_x\|_{L^2_{per}(Q)} + \|u_{yy}\|_{L^2_{per}(Q)} + \|u_y\|_{L^{\infty,2}_{per}(Q)} \leq C_{f,X}\|\varphi_y\|_{L^2_{per}(\mathbb{R})}.$$

1.6 The Homogenized Problem

Consider the homogenized problem (1.6)–(1.8). We have the following from the results of Section 1.3.

Theorem 6 *Assume that conditions (1.14), (1.22)–(1.24) hold and that there exist limits (1.9). Then there exists a unique solution* $u_0 \in H^{1,2}_{per}(Q)$ *of problem (1.6)–(1.8) and it satisfies the estimates*

$$\|u_0\|_{C_{per}(\overline{Q})} \leq \|\varphi\|_{C_{per}(\mathbb{R})},$$
$$\|u_{0x}\|_{L^2_{per}(Q)} + \|u_{0yy}\|_{L^2_{per}(Q)} + \|u_{0y}\|_{L^{\infty,2}_{per}(Q)} \leq C_{f,X}\|\varphi_y\|_{L^2_{per}(\mathbb{R})}.$$

If in addition conditions (1.18) hold then $u_{0y} \in H^{1,2}_{per}(Q)$ *and the following estimate holds:*

$$\|u_{0yx}\|_{L^2_{per}(Q)} + \|u_{0yyy}\|_{L^2_{per}(Q)} + \|u_{0yy}\|_{L^{\infty,2}_{per}(Q)} \leq C_{f,X}\|\varphi_{yy}\|_{L^2_{per}(\mathbb{R})}.$$

1.7 Error Estimates for the Asymptotic Approximation

Theorem 7 *Assume that conditions (1.14), (1.22)–(1.24) are satisfied and that limits (1.9) exist. Assume that the following estimates hold:*

$$\|\widetilde{\beta}\|_{L^\infty(0,\xi)} \leq A\xi^d, \quad \|\widetilde{\alpha}\|_{L^2(0,\xi)} \leq A\xi^{d+1/2}, \quad \|\widetilde{\nu}\|_{L^2(0,\xi)} \leq A\xi^{d+1/2} \quad \forall \xi \in \mathbb{R}^+$$

with some constants $A > 0$ *and* $d \in [0, 1)$ *and that there exists* $r \in (2, \infty]$ *such that*

$$\|\alpha\|_{L^r(0,\xi)} \leq \kappa_2 \xi^{1/r} \quad \forall \xi \in \mathbb{R}^+.$$

Then the following estimate holds:

$$\|u - u_0\|_{L^2_{per}(Q)} \leq AC_{f,X,r}\|\varphi_y\|_{L^2_{per}(\mathbb{R})}\, \delta^{1-d}. \tag{1.25}$$

Theorem 8 *Assume that conditions (1.18), (1.22)–(1.24) are satisfied and that limits (1.9) exist. Assume that the following estimates hold:*

$$\|\widetilde{\beta}\|_{L^\infty(0,\xi)} \leq A\xi^d, \quad \|\widetilde{\alpha}\|_{L^\infty(0,\xi)} \leq A\xi^d, \quad \|\widetilde{\nu}\|_{L^\infty(0,\xi)} \leq A\xi^d \quad \forall \xi \in \mathbb{R}^+$$

with some constants $A > 0$ *and* $d \in [0, 1)$.

Then the following error estimate holds:

$$\|u - u_0\|_{V_{per}(Q)} \leq AC_{f,X}\|\varphi_{yy}\|_{L^2_{per}(Q)}\delta^{1-d}. \tag{1.26}$$

Proofs are based on Theorems 3, 4, and 5.

Remark 1. If coefficients β, α, ν are δ-periodical functions then $d = 0$ and estimates (1.25), (1.26) have the following form:

$$\|u - u_0\|_{L^2_{per}(Q)} \leq AC_{f,X,r}\|\varphi_y\|_{L^2_{per}(\mathbb{R})}\, \delta,$$
$$\|u - u_0\|_{V_{per}(Q)} \leq AC_{f,X}\|\varphi_{yy}\|_{L^2_{per}(Q)}\delta.$$

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2

Geometric Versus Spectral Convergence for the Neumann Laplacian under Exterior Perturbations of the Domain

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2.1 Introduction

This chapter is concerned with the behavior of the eigenvalues and eigenfunctions of the Laplace operator in bounded domains when the domain undergoes a perturbation. It is well known that if the boundary condition that we are imposing is of Dirichlet type, the kind of perturbations that we may allow in order to obtain the continuity of the spectra is much broader than in the case of a Neumann boundary condition. This is explicitly stated in the pioneer work of Courant and Hilbert [CoHi53], and it has been subsequently clarified in many works, see [BaVy65, Ar97, Da03] and the references therein among others. See also [HeA06] for a general text on different properties of eigenvalues and [HeD05] for a study on the behavior of eigenvalues and in general partial differential equations when the domain is perturbed.

In particular, with a Dirichlet boundary condition we may consider the case where the fixed domain is a bounded "smooth" domain $\Omega_0 \subset \mathbb{R}^N$, $N \geq 2$, and the perturbed domain is Ω_ϵ in such a way that $\Omega_0 \subset \Omega_\epsilon$, that is, we consider exterior perturbation of the domain. We may have perturbations of this type where $|\Omega_\epsilon \setminus \Omega_0| \geq \eta$ for some fixed $\eta > 0$, and still we have the convergence of the eigenvalues and eigenfunctions. Moreover, we may even have the case $|\Omega_\epsilon \setminus \Omega_0| \to +\infty$, and still we have the convergence of the eigenvalues and eigenfunctions.

To obtain an example of this situation is not too difficult. If we consider, for instance, $\Omega \subset \mathbb{R}^2$, given by $\Omega_0 = (0,1) \times (-1,0)$ and

$$\Omega_\epsilon(a) = \{(x,y) : 0 < x < 1, -1 < y < a(1 + \sin(x/\epsilon))\} \supset \Omega_0$$

where $a > 0$ is fixed, we can easily see that the eigenvalues and eigenfunctions of the Laplace operator with Dirichlet boundary condition in Ω_ϵ converge to the ones in Ω_0. Moreover, $|\Omega_\epsilon| = |\Omega_0| + \int_0^1 a(1 + \sin(x/\epsilon))dx \sim |\Omega_0| + a$

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for ϵ small enough. Moreover, it is not difficult to modify the example above choosing the constant a dependent with respect to ϵ in such a way that $a(\epsilon) \to +\infty$ and such that the eigenvalues and eigenfunctions in $\Omega_\epsilon(a(\epsilon))$ still converge to the ones in Ω_0 and $|\Omega_\epsilon(a(\epsilon)) \setminus \Omega_0| \to +\infty$. This example shows that the class of perturbations that we may allow to get the "spectral convergence" of the Dirichlet Laplacian is very broad and that knowing that the eigenvalues and eigenfunctions of the Dirichlet Laplacian converge does not have many "geometrical" restrictions for the domains.

The case of the Neumann boundary condition is much more subtle. As a matter of fact, for the situation depicted above, it is not true that the spectra converge. So we ask ourselves the following questions: if we have a domain Ω_0 and consider a perturbation of it given by $\Omega_0 \subset \Omega_\epsilon$, where we assume that all the domains are smooth and bounded although not necessarily uniformly bounded on the parameter ϵ, then if we have the convergence of the eigenvalues and eigenfunctions,

(Q1) should it be true that $|\Omega_\epsilon \setminus \Omega_0| \overset{\epsilon\to 0}{\longrightarrow} 0$?

(Q2) should it be true that $\mathrm{dist}(\Omega_\epsilon, \Omega_0) = \sup_{x\in\Omega_\epsilon} \mathrm{dist}(x, \Omega_0) \overset{\epsilon\to 0}{\longrightarrow} 0$?

We will see that the answer to the first question is Yes and, surprisingly, the answer to the second one is No.

Observe that, as the example above shows, the answer to both questions for the case of the Dirichlet boundary condition is No.

In Section 2.2 we recall a result from [Ar95, ArCa04] which provides a necessary and sufficient condition for the convergence of eigenvalues and eigenfunctions when the domain is perturbed. In Section 2.3 we provide an answer to question **(Q1)**, and in Section 2.4 we provide an answer to question **(Q2)**.

2.2 Characterization of the Spectral Convergence of the Neumann Laplacian

In this section we give a necessary and sufficient condition for the convergence of the eigenvalues and eigenfunctions of the Laplace operator with Neumann boundary conditions. We refer to [Ar95] and [ArCa04] for a general result in this direction, in even a more general context than the one in this chapter. In our particular case, we will consider the following situation: let Ω_0 be a fixed bounded smooth (Lipschitz is enough) open set in $\mathbb{R}^N$ with $N \geq 2$ and let Ω_ϵ be a family of domains such that, for each fixed $0 < \epsilon \leq \epsilon_0$, Ω_ϵ is bounded and smooth with $\Omega_0 \subset \Omega_\epsilon$.

Let us define now what we mean by the spectral convergence. For $0 \leq \epsilon \leq \epsilon_0$, we denote by $\{\lambda_n^\epsilon\}_{n=1}^\infty$ the sequence of eigenvalues of the Neumann Laplacian in Ω_ϵ, always ordered and counting its multiplicity, and we denote by $\{\phi_n^\epsilon\}_{n=1}^\infty$ a corresponding set of orthonormal eigenfunctions in Ω_ϵ. Also, since we are considering domains which vary with the parameter ϵ, and we will need to compare functions defined in Ω_0 and in Ω_ϵ, we introduce the following space $H^1_\epsilon = H^1(\Omega_0) \oplus H^1(\Omega_\epsilon \setminus \bar{\Omega}_0)$, that is, $\chi \in H^1_\epsilon$ if $\chi_{|\Omega_0} \in H^1(\Omega_0)$ and $\chi_{|(\Omega_\epsilon \setminus \bar{\Omega}_0)} \in H^1(\Omega_\epsilon \setminus \bar{\Omega}_0)$, with the norm

$$\|\chi\|^2_{H^1_\epsilon} = \|\chi\|^2_{H^1(\Omega_0)} + \|\chi\|^2_{H^1(\Omega_\epsilon \setminus \bar{\Omega}_0)} .$$

We have that $H^1(\Omega_\epsilon) \hookrightarrow H^1_\epsilon$ and in a natural way we have that if $\chi \in H^1(\Omega_0)$ via the extension by zero outside Ω_0 we have $\chi \in H^1_\epsilon$. Hence, with certain abuse of notation we may say that if $\chi_\epsilon \in H^1_\epsilon$, $0 \le \epsilon \le \epsilon_0$, then $\chi_\epsilon \stackrel{\epsilon \to 0}{\longrightarrow} \chi_0$ in H^1_ϵ if $\|\chi_\epsilon - \chi_0\|_{H^1(\Omega_0)} + \|\chi_\epsilon\|_{H^1(\Omega_\epsilon \setminus \Omega_0)} \stackrel{\epsilon \to 0}{\longrightarrow} 0$.

Definition 1. *We will say that the family of domains Ω_ϵ converges spectrally to Ω_0 as $\epsilon \to 0$ if the eigenvalues and eigenprojectors of the Neumann Laplacian behave continuously at $\epsilon = 0$. That is, for any fixed $n \in \mathbb{N}$ we have that $\lambda^\epsilon_n \to \lambda^0_n$ as $\epsilon \to 0$, and for each $n \in \mathbb{N}$ such that $\lambda^0_n < \lambda^0_{n+1}$ the spectral projections $P^\epsilon_n : L^2(\mathbb{R}^N) \to H^1(\Omega_\epsilon)$, $P^\epsilon_n(\psi) = \sum_{i=1}^n (\phi^\epsilon_i, \psi)_{L^2(\Omega_\epsilon)} \phi^\epsilon_i$, satisfy*

$$\sup\{\|P^\epsilon_n(\psi) - P^0_n(\psi)\|_{H^1_\epsilon}, \psi \in L^2(\mathbb{R}^N), \|\psi\|_{L^2(\mathbb{R}^N)} = 1\} \stackrel{\epsilon \to 0}{\longrightarrow} 0 .$$

The convergence of the spectral projections is equivalent to the following: for each sequence $\epsilon_k \to 0$ there exists a subsequence, that we denote again by ϵ_k, and a complete system of orthonormal eigenfunctions of the limiting problem $\{\phi^0_n\}_{n=1}^\infty$ such that $\|\phi^{\epsilon_k}_n - \phi^0_n\|_{H^1_{\epsilon_k}} \to 0$ as $k \to \infty$.

In order to write down the characterization, we need to consider the following quantity:

$$\tau_\epsilon = \min_{\substack{\phi \in H^1(\Omega_\epsilon) \\ \phi = 0 \text{ in } \Omega_0}} \frac{\displaystyle\int_{\Omega_\epsilon} |\nabla \phi|^2}{\displaystyle\int_{\Omega_\epsilon} |\phi|^2} . \tag{2.1}$$

Observe that τ_ϵ is the first eigenvalue of the following problem with a combination of Dirichlet and Neumann boundary conditions:

$$\begin{cases} -\Delta u = \tau u , & \Omega_\epsilon \setminus \bar{\Omega}_0 , \\ u = 0 , & \partial\Omega_0 , \\ \dfrac{\partial u}{\partial n} = 0 , & \partial\Omega_\epsilon \setminus \partial\Omega_0 . \end{cases}$$

We can prove the following assertion.

Proposition 1. *A necessary and sufficient condition for the spectral convergence of Ω_ϵ to Ω_0 is*

$$\tau_\epsilon \stackrel{\epsilon \to 0}{\longrightarrow} +\infty . \tag{2.2}$$

We refer to [Ar95] and [ArCa04] for a proof of this result.

Remark 1. The fact that $\Omega_0 \subset \Omega_\epsilon$ can be relaxed. It is enough asking that for each compact set $K \subset \Omega_0$ there exists $\epsilon(K)$ such that $K \subset \Omega_\epsilon$ for $0 < \epsilon \leq \epsilon(K)$, see [ArCa04].

2.3 Measure Convergence of the Domains

In this section we provide an answer to the first question. Observe that in Proposition 1 we do not require that $|\Omega_\epsilon \setminus \Omega_0| \stackrel{\epsilon \to 0}{\longrightarrow} 0$. However, we have the following.

Corollary 1. *In the situation above if Ω_ϵ converges spectrally to Ω_0, then necessarily $|\Omega_\epsilon \setminus \Omega_0| \stackrel{\epsilon \to 0}{\longrightarrow} 0$.*

Proof. This result is proved in [ArCa04], but for the sake of completeness and since it is a simple proof, we include it here.

If this were not true, then we would have a positive $\eta > 0$ and a sequence $\epsilon_k \to 0$ such that $|\Omega_{\epsilon_k} \setminus \Omega_0| \geq \eta$. Let $\rho = \rho(\eta)$ be a small number such that $|\{x \in \mathbb{R}^N \setminus \Omega_0, \operatorname{dist}(x, \Omega_0) \leq \rho\}| \leq \eta/2$. This implies that $|\{x \in \Omega_{\epsilon_k}, \operatorname{dist}(x, \Omega_0) \geq \rho\}| \geq \eta/2$. Let us construct a smooth function γ with $\gamma = 0$ in Ω_0, and $\gamma(x) = 1$ for $x \in \mathbb{R}^N \setminus \Omega_0$ with $\operatorname{dist}(x, \Omega_0) \geq \rho$. Then obviously $\gamma \in H^1(\Omega_{\epsilon_k})$ with $\|\nabla\gamma\|_{L^2(\Omega_{\epsilon_k})} \leq C$ and $\|\gamma\|_{L^2(\Omega_{\epsilon_k})} \geq (\eta/2)^{\frac{1}{2}}$. This implies that τ_{ϵ_k} is bounded. Hence, it is not true that $\tau_\epsilon \stackrel{\epsilon \to 0}{\longrightarrow} +\infty$ and, therefore, from Proposition 1, we do not obtain the spectral convergence.

In particular, this result implies that the answer to question **(Q1)** is affirmative. That is, if we have the convergence of Neumann eigenvalues and eigenfunctions, necessarily we have that $|\Omega_\epsilon \setminus \Omega_0| \stackrel{\epsilon \to 0}{\longrightarrow} 0$.

2.4 Distance Convergence of the Domains

In this section we will provide an answer to question **(Q2)**, and we will see that the answer is No. We will prove this by constructing an example of a fixed domain Ω_0 and a sequence of domains Ω_ϵ with $\Omega_0 \subset \Omega_\epsilon$ with the property that $\operatorname{dist}(\Omega_\epsilon, \Omega_0)$ does not converges to 0, but the eigenvalues and eigenfunctions of the Laplace operator with Neumann boundary conditions in Ω_ϵ converge to the ones in Ω_0, see Definition 1.

As a matter of fact, in [ArCa04, Section 5.2] a very particular example of a dumbbell domain (two disconnected domains joined by a thin channel) is provided so that the eigenvalues from the dumbbell converge to the eigenvalues of the two disconnected domains and no spectral contribution from the channel is observed. In this chapter we will obtain a family of channels for which the same phenomenon occurs, see Corollary 2, and we will provide a proof different from the one given in [ArCa04].

Let us consider a fixed domain $\Omega_0 \subset \mathbb{R}^N$ which satisfies $\Omega_0 \subset \{x \in \mathbb{R}^N,\, x_1 < 0\}$ and such that

$$\Omega_0 \cap \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1}, -1 < x_1 < 1,\, |x'| \leq \rho\} \\ = \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1}, -1 < x_1 < 0,\, |x'| \leq \rho\}$$

for some fixed $\rho > 0$.

We will construct Ω_ϵ as $\Omega_\epsilon = \mathrm{int}(\bar{\Omega}_0 \cup \bar{R}_\epsilon)$, where R_ϵ is given as follows:

$$R_\epsilon = \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} :\, 0 < x_1 < L,\, |x'| < g_\epsilon(x_1)\}, \tag{2.3}$$

where the function g_ϵ will be chosen so that $g_\epsilon > 0$, $g_\epsilon \in C^1([0, L])$, and $g_\epsilon \to 0$ uniformly on $[0, L]$; see Figure 2.1. For the sake of notation, we denote by $\Gamma_0^\epsilon = \partial R_\epsilon \cap \{x_1 = 0\}$ and $\Gamma_L^\epsilon = \partial R_\epsilon \cap \{x_1 = L\}$.

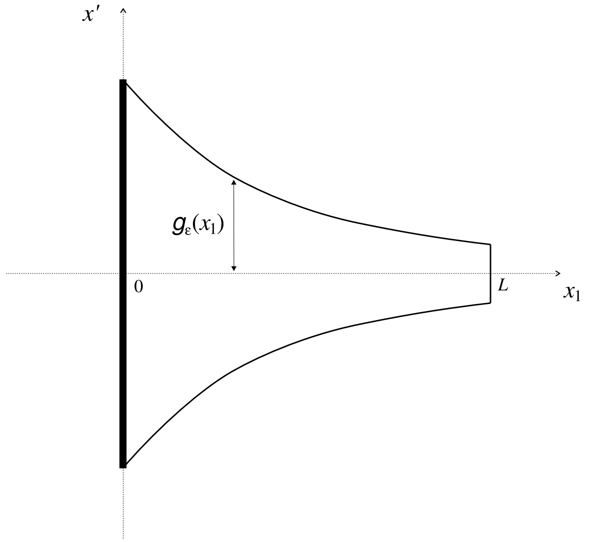

Fig. 2.1. The exterior perturbation R_ϵ. The thick line refers to the supplementary Dirichlet condition in the problem (2.4), while Neumann boundary conditions are imposed elsewhere.

We refer to [Ra95] for a general reference on the behavior of solutions of partial differential equations on thin domains. See also the recent survey [Gr08] for a study on the spectrum of the Laplacian on thin tubes in various settings, and for many related references.

Observe that if L is fixed, then $\mathrm{dist}(\Omega_\epsilon, \Omega_0) = L$ for each $0 < \epsilon \leq \epsilon_0$. Moreover, we will show that for certain choices of g_ϵ we obtain the spectral convergence of the Laplace operator. To prove this result, we use Proposition 1 and show that $\tau_\epsilon \to +\infty$. Notice that τ_ϵ, defined in (2.1) is the first eigenvalue of

$$\begin{cases} -\Delta u = \tau u\,, & R_\epsilon\,, \\ u = 0\,, & \Gamma_0^\epsilon\,, \\ \dfrac{\partial u}{\partial n} = 0\,, & \partial R_\epsilon \setminus \Gamma_0^\epsilon\,. \end{cases} \tag{2.4}$$

Since we have Neumann boundary conditions on the lateral boundary of R_ϵ, there clearly exist profiles of g_ϵ for which τ_ϵ remains uniformly bounded as $\epsilon \to 0$. In fact, a simple trial-function argument shows that $\tau_\epsilon \leq \pi^2/(2L)^2$ whenever $g_\epsilon(s) \geq g_\epsilon(0)$ for every $s \in [0, L]$. The idea to get $\tau_\epsilon \to +\infty$ consists in choosing a rapidly decreasing function $s \mapsto g_\epsilon(s)$, which enables one to get a large contribution to τ_ϵ coming from the longitudinal energy due to the approaching Dirichlet and Neumann boundary conditions in the limit $\epsilon \to 0$. Let us notice that a similar trick to employ the repulsive contribution of such a combination of the boundary conditions has been used recently in [KoKr08] to establish a Hardy-type inequality in a waveguide; see also [Kr09] for eigenvalue asymptotics in narrow curved strips with combined Dirichlet and Neumann boundary conditions. In our case, we are able to show the following.

Proposition 2. *With the notation above, for any function $\gamma \in C^2([0, L])$ satisfying*

$$0 < \alpha_0 \leq \gamma \leq \alpha_1 < 1, \qquad \dot\gamma(L) \leq 0, \qquad \text{and} \qquad \ddot\gamma \geq \alpha_2 > 0 \tag{2.5}$$

for some positive numbers α_0, α_1, and α_2, if we define $g_\epsilon = \gamma^{1/\epsilon}$ we have that $\tau_\epsilon \xrightarrow{\epsilon \to 0} +\infty$.

In particular, applying Proposition 1 we obtain the convergence of the eigenvalues and eigenfunctions of the Neumann Laplacian in Ω_ϵ to the ones in Ω_0.

Remark 2. Observe that a function γ satisfying (2.5) necessarily satisfies $\dot\gamma(s) < 0$ for $0 \leq s < L$. Hence, the function γ is decreasing.

Proof. Since τ_ϵ is given by minimization of the Rayleigh quotient,

$$\tau_\epsilon = \inf_{\substack{\phi \in H^1(R_\epsilon) \\ \phi = 0 \text{ in } \Gamma_0^\epsilon}} \frac{\int_{R_\epsilon} |\nabla \phi|^2}{\int_{R_\epsilon} |\phi|^2},$$

we analyze the integral $\int_{R_\epsilon} |\nabla \phi|^2$ for a smooth real-valued function ϕ with $\phi = 0$ in a neighborhood of Γ_0^ϵ. We have

$$\int_{R_\epsilon} |\nabla \phi|^2 = \int_0^L \int_{|x'| < g_\epsilon(x_1)} \left(|\phi_{x_1}|^2 + |\nabla_{x'} \phi|^2\right) dx' dx_1.$$

Considering the change of variables $x_1 = y_1$, $x' = g_\epsilon(y_1)y'$ which transforms $(x_1, x') \in R_\epsilon$ into $(y_1, y') \in Q$, where Q is the cylinder $Q = \{(y_1, y') : 0 < y_1 < L, |y'| < 1\}$ and performing this change of variables in the integral above, elementary calculations show that

$$\int_{R_\epsilon} |\nabla\phi|^2 = \int_Q \left[\left(\varphi_{y_1} - \frac{\dot{g}_\epsilon}{g_\epsilon} \sum_{i=2}^N y_i \varphi_{y_i} \right)^2 + \frac{1}{g_\epsilon^2} \sum_{i=2}^N |\varphi_{y_i}|^2 \right] g_\epsilon^{N-1} dy,$$

where $\varphi(y) = \phi\big(y_1, g_\epsilon(y_1)y'\big)$.

Writing the above expression in terms of the new function $\psi(y) = g_\epsilon(y_1)^{\frac{N-1}{2}} \varphi(y)$ so that

$$\begin{aligned} g_\epsilon^{(N-1)/2} \varphi_{y_i} &= \psi_{y_i}\,, \qquad i = 2, \dots, N\,, \\ g_\epsilon^{(N-1)/2} \varphi_{y_1} &= -\frac{N-1}{2} \frac{\dot{g}_\epsilon}{g_\epsilon} \psi + \psi_{y_1}\,, \end{aligned}$$

we get

$$\begin{aligned} &\int_{R_\epsilon} |\nabla\phi|^2 \\ &\quad = \int_Q \left[\left(-\frac{N-1}{2} \frac{\dot{g}_\epsilon}{g_\epsilon} \psi + \psi_{y_1} - \frac{\dot{g}_\epsilon}{g_\epsilon} \sum_{i=2}^N y_i \psi_{y_i} \right)^2 + \frac{1}{g_\epsilon^2} \sum_{i=2}^N |\psi_{y_i}|^2 \right] dy \\ &\quad = \int_Q \left[\left(-\frac{N-1}{2} \frac{\dot{g}_\epsilon}{g_\epsilon} \psi \right)^2 + \left(\psi_{y_1} - \frac{\dot{g}_\epsilon}{g_\epsilon} \sum_{i=2}^N y_i \psi_{y_i} \right)^2 - (N-1) \frac{\dot{g}_\epsilon}{g_\epsilon} \psi \psi_{y_1} \right. \\ &\qquad\qquad \left. + \ (N-1) \frac{\dot{g}_\epsilon^2}{g_\epsilon^2} \sum_{i=2}^N y_i \psi_{y_i} \psi + \frac{1}{g_\epsilon^2} \sum_{i=2}^N |\psi_{y_i}|^2 \right] dy, \\ &\quad \geq \int_Q \left[\left(\frac{N-1}{2} \right)^2 \frac{\dot{g}_\epsilon^2}{g_\epsilon^2} \psi^2 - (N-1) \frac{\dot{g}_\epsilon}{g_\epsilon} \psi \psi_{y_1} \right. \\ &\qquad\qquad \left. + (N-1) \frac{\dot{g}^2}{g_\epsilon^2} \sum_{i=2}^N y_i \psi_{y_i} \psi + \frac{1}{g_\epsilon^2} \sum_{i=2}^N \psi_{y_i}^2 \right] dy \end{aligned}$$

where we have used $(\psi_{y_1} - \sum_{i=2}^N y_i \psi_{y_i} \frac{\dot{g}_\epsilon}{g_\epsilon})^2 \geq 0$. Via integration by parts in the second and third terms above, we get

$$\begin{aligned} &\int_Q -(N-1) \frac{\dot{g}_\epsilon}{g_\epsilon} \psi \psi_{y_1} dy = \int_{|y'|<1} \int_0^L -(N-1) \frac{\dot{g}_\epsilon}{2g_\epsilon} (\psi^2)_{y_1} dy_1 dy' \\ &\quad = \int_{|y'|<1} \left(-\left[(N-1) \frac{\dot{g}_\epsilon}{2g_\epsilon} \psi^2 \right]_{y_1=0}^{y_1=L} + \int_0^L (N-1) \left(\frac{\dot{g}_\epsilon}{2g_\epsilon} \right)' \psi^2 dy_1 \right) dy' \\ &\quad = -\int_{|y'|<1} (N-1) \frac{\dot{g}_\epsilon(L)}{2g_\epsilon(L)} \psi^2(L, y') dy' + \int_Q \frac{N-1}{2} \left(\frac{\ddot{g}_\epsilon}{g_\epsilon} - \frac{\dot{g}_\epsilon^2}{g_\epsilon^2} \right) \psi^2 dy \end{aligned}$$

and

$$\int_Q (N-1)\frac{\dot g_\epsilon^2}{g_\epsilon^2}\sum_{i=2}^N y_i\psi_{y_i}\psi dy = \int_0^L (N-1)\frac{\dot g_\epsilon^2}{g_\epsilon^2}\sum_{i=2}^N \int_{|y'|<1} y_i\frac{1}{2}(\psi^2)_{y_i}dy'dy_1$$
$$= \int_0^L \frac{N-1}{2}\frac{\dot g_\epsilon^2}{g_\epsilon^2}\left(\int_{|y'|=1}\psi^2 - (N-1)\int_{|y'|<1}\psi^2 dy'\right)dy_1 .$$

Hence, if we require that $\dot g_\epsilon(L) \le 0$, we have

$$\int_{R_\epsilon}|\nabla\phi|^2 \ge \int_Q\left[\frac{N-1}{2}\frac{\ddot g_\epsilon}{g_\epsilon} - \left(\left(\frac{N-1}{2}\right)^2 + \frac{N-1}{2}\right)\frac{\dot g_\epsilon^2}{g_\epsilon^2}\right]\psi^2 dy$$
$$+\int_0^L \frac{N-1}{2}\frac{\dot g_\epsilon^2}{g_\epsilon^2}\left(\int_{|y'|=1}\psi^2 dy'\right)dy_1 + \int_Q \frac{1}{g_\epsilon^2}\sum_{i=2}^N \psi_{y_i}^2 dy . \tag{2.6}$$

The last two terms in this expression can be written as

$$\int_0^L \frac{1}{g_\epsilon^2(y_1)}\left(\int_{|y'|\le 1}|\nabla_{y'}\psi|^2 + \frac{N-1}{2}\,\dot g_\epsilon^2(y_1)\int_{|y'|=1}\psi^2\right)dy_1$$

and we have that

$$\int_{|y'|\le 1}|\nabla_{y'}\psi|^2 + \frac{N-1}{2}\,\dot g_\epsilon^2\int_{|y'|=1}\psi^2 \ge \rho\int_{|y'|\le 1}\psi^2$$

with $\rho = \rho(y_1)$ being the first eigenvalue of the problem

$$\begin{cases} -\Delta_{y'}\psi = \rho\psi\,, & |y'|<1\,,\\ \dfrac{\partial\psi}{\partial n} + \dfrac{N-1}{2}\,\dot g_\epsilon^2(y_1)\psi = 0\,, & |y'| = 1\,,\end{cases}$$

where n denotes the outward unit normal vector field to the $(N-2)$-dimensional unit sphere $S_1 = \{y' \in \mathbb{R}^{N-1} : |y'| = 1\}$.

We claim that if we denote by $\lambda(\eta)$ the first eigenvalue of

$$\begin{cases} -\Delta_{y'}\psi = \lambda\psi\,, & |y'|<1\,,\\ \dfrac{\partial\psi}{\partial n} + \eta\psi = 0\,, & |y'| = 1\,,\end{cases}$$

we have that $\frac{\lambda(\eta)}{\eta} \to \frac{|S_1|}{|B_1|}$ as $\eta \to 0$, where B_1 is the $(N-1)$-dimensional unit ball and S_1 its surface, which satisfy $|S_1| = (N-1)|B_1|$. As a matter of fact, by a standard continuity result, we know that $\lambda(\eta) \to 0$ and its eigenfunction ψ_η, which is radially symmetric, converges to the constant function $1/\sqrt{|B_1|}$, which is the first eigenfunction of the Neumann eigenvalue problem. But

$$\lambda(\eta) = \int_{B_1}|\nabla_{y'}\psi_\eta|^2 + \eta\int_{S_1}|\psi_\eta|^2 \ge \eta\int_{S_1}|\psi_\eta|^2,$$

which implies that

$$\frac{\lambda(\eta)}{\eta} \geq \int_{S_1} |\psi_\eta|^2 \to \frac{|S_1|}{|B_1|}.$$

Moreover, using $\psi = 1/\sqrt{|B_1|}$ as a test function in the Rayleigh quotient for $\lambda(\eta)$, we immediately obtain $\lambda(\eta) \leq \eta\frac{|S_1|}{|B_1|}$. This proves our claim. In particular, given $\delta > 0$ small, we can choose $\eta_0 = \eta_0(\delta)$ such that $\lambda(\eta) > (N-1-\delta)\eta$ for $0 < \eta \leq \eta_0$.

Therefore, if we choose the function g_ϵ such that $\dot{g}_\epsilon(y_1) \to 0$ uniformly in $y_1 \in [0, L]$, we have that $\rho(y_1) \geq \frac{(N-1)(N-1-\delta)}{2}\, \dot{g}_\epsilon{}^2(y_1)$ for ϵ small enough.

Hence,

$$\begin{aligned} \int_{R_\epsilon} |\nabla\phi|^2 \;&\geq\; \int_Q \left\{ \frac{N-1}{2}\frac{\ddot{g}_\epsilon}{g_\epsilon} - \left[\left(\frac{N-1}{2}\right)^2 \right.\right. \\ &\qquad \left.\left. - \frac{(N-1)(N-1-\delta)}{2} + \frac{N-1}{2} \right] \frac{\dot{g}_\epsilon{}^2}{g_\epsilon^2} \right\} \psi^2 dy \\ &= \frac{N-1}{2} \int_Q \left\{ \frac{\ddot{g}_\epsilon}{g_\epsilon} - \left[\frac{N-1}{2} - (N-1-\delta) + 1 \right] \frac{\dot{g}_\epsilon^2}{g_\epsilon^2} \right\} \psi^2 dy \end{aligned}$$

and observe that the number $\kappa = \frac{N-1}{2} - (N-1-\delta) + 1$ is strictly less than one for all values of $N \geq 2$ choosing a fixed and small $\delta > 0$. If we denote

$$m_\epsilon = \inf_{0 \leq y_1 \leq L} \left(\frac{\ddot{g}_\epsilon}{g_\epsilon} - \kappa\, \frac{\dot{g}_\epsilon^2}{g_\epsilon^2} \right),$$

then

$$\int_{R_\epsilon} |\nabla\phi|^2 \geq \frac{N-1}{2}\, m_\epsilon \int_Q \psi^2 = \frac{N-1}{2}\, m_\epsilon \int_{R_\epsilon} \phi^2 .$$

Consequently, $\tau_\epsilon \geq \frac{N-1}{2}\, m_\epsilon$.

Let us see that we can make a choice of the family of functions g_ϵ, satisfying the two previous conditions we have imposed, that is, $\dot{g}_\epsilon(L) \leq 0$ and $\dot{g}_\epsilon(y_1) \to 0$ uniformly in $0 \leq y_1 \leq L$ such that $m_\epsilon \to +\infty$ as $\epsilon \to 0$.

Let us choose a function $\gamma \in C^2([0, L])$ satisfying (2.5) and let $g_\epsilon = \gamma^{1/\epsilon}$. Then, we have

$$\dot{g}_\epsilon = \frac{1}{\epsilon}\gamma^{\frac{1}{\epsilon}-1}\dot{\gamma}, \qquad \ddot{g}_\epsilon = \frac{1}{\epsilon}(\frac{1}{\epsilon} - 1)\gamma^{\frac{1}{\epsilon}-2}\dot{\gamma}^2 + \frac{1}{\epsilon}\gamma^{\frac{1}{\epsilon}-1}\ddot{\gamma},$$

and simple calculations show that

$$\frac{\ddot{g}_\epsilon}{g_\epsilon} - \kappa\, \frac{\dot{g}_\epsilon^2}{g_\epsilon^2} = \left[\frac{1}{\epsilon}(\frac{1}{\epsilon} - 1) - \kappa \left(\frac{1}{\epsilon}\right)^2 \right] \left(\frac{\dot{\gamma}}{\gamma}\right)^2 + \frac{\ddot{\gamma}}{\epsilon\gamma} \geq \frac{\alpha_2}{\alpha_0}\frac{1}{\epsilon}$$

for $\epsilon > 0$ small enough so that $\frac{1}{\epsilon}(\frac{1}{\epsilon} - 1) - \kappa\left(\frac{1}{\epsilon}\right)^2 \geq 0$. This shows that $m_\epsilon \to +\infty$ and it proves the proposition.

Remark 3. Now that we have been able to construct a thin domain R_ϵ as in (2.3) such that $\tau_\epsilon \stackrel{\epsilon\to 0}{\longrightarrow} +\infty$, we can construct another thin domain $\tilde{R}_\epsilon$ such that its "length" goes to infinity, its width goes to zero, and still $\tilde{\tau}_\epsilon \stackrel{\epsilon\to 0}{\longrightarrow} +\infty$, where $\tilde{\tau}_\epsilon$ is the first eigenvalue of (2.4) in $\tilde{R}_\epsilon$ instead of R_ϵ.

For this, let R_ϵ be a thin domain constructed as in Proposition 2 and let ρ_ϵ be a sequence with $\rho_\epsilon \to +\infty$ such that $\frac{\tau_\epsilon}{\rho_\epsilon^2} \to +\infty$ and $\alpha_1^{1/\epsilon}\rho_\epsilon \to 0$. Define $\tilde{R}_\epsilon = \rho_\epsilon R_\epsilon$, that is,

$$\tilde{R}_\epsilon = \{(x_1, x') : 0 < x_1 < \rho_\epsilon L, \, |x'| < \rho_\epsilon g_\epsilon(x_1)\} ,$$

then $0 < \rho_\epsilon g_\epsilon(x_1) \le \alpha_1^{1/\epsilon}\rho_\epsilon \stackrel{\epsilon\to 0}{\longrightarrow} 0$ and $\tilde{\tau}_\epsilon = \frac{\tau_\epsilon}{\rho_\epsilon^2} \stackrel{\epsilon\to 0}{\longrightarrow} +\infty$.

Observe that if we also require a Dirichlet boundary condition in Γ_L^ϵ, we can relax the conditions on γ in Proposition 2 and in particular the condition $\dot{\gamma}(L) \le 0$ can be dropped. Hence, we can show the following.

Corollary 2. *With the notation above, for any function $\gamma \in C^2([0, L])$ satisfying*

$$0 < \alpha_0 \le \gamma \le \alpha_1 < 1, \qquad \textit{and} \qquad \ddot{\gamma} \ge \alpha_2 > 0$$

for some positive numbers α_0, α_1, and α_2, if we define $g_\epsilon = \gamma^{1/\epsilon}$ we have $\tilde{\tau}_\epsilon \stackrel{\epsilon\to 0}{\longrightarrow} +\infty$, where $\tilde{\tau}\epsilon$ is the first eigenvalue of

$$\begin{cases} -\Delta u = \tau u \, , & R_\epsilon \, , \\ u = 0 \, , & \Gamma_0^\epsilon \cup \Gamma_L^\epsilon \, , \\ \dfrac{\partial u}{\partial n} = 0 \, , & \partial R_\epsilon \setminus (\Gamma_0^\epsilon \cup \Gamma_L^\epsilon) \, . \end{cases}$$

Proof. This follows easily by a Neumann bracketing argument. More precisely, from the hypotheses, $\dot{\gamma}$ is a strictly increasing function. Hence, either γ is strictly monotone in $(0, L)$, or there exists a unique $L^* \in (0, L)$ such that $\dot{\gamma}(L^*) = 0$.

In the first case, if γ is decreasing (respectively increasing) we substitute the Dirichlet boundary condition at Γ_L^ϵ (respectively at Γ_0^ϵ) by a Neumann one. Then the new eigenvalue problem gives rise to τ_ϵ defined exactly in the same way as (2.4) (modulo possibly a mirroring of R_ϵ), and we have $\tilde{\tau}_\epsilon \ge \tau_\epsilon \to +\infty$ as $\epsilon \to 0$.

In the second case, we cut the domain R_ϵ in two domains $R_\epsilon^0 = R_\epsilon \cap \{0 < x_1 < L^*\}$, $R_\epsilon^1 = R_\epsilon \cap \{L^* < x_1 < L\}$. We know that $\tilde{\tau}_\epsilon \ge \inf\{\tau_\epsilon^0, \tau_\epsilon^1\}$, where τ_ϵ^0 and τ_ϵ^1 are the corresponding eigenvalues in R_ϵ^0 and R_ϵ^1 with a Neumann boundary condition imposed at the newly created boundary $R_\epsilon \cap \{x_1 = L^*\}$ on both domains. In both domains we can apply Proposition 2 as in the first case so that $\tau_\epsilon^0, \tau_\epsilon^1 \stackrel{\epsilon\to 0}{\longrightarrow} +\infty$, which implies $\tilde{\tau}_\epsilon \to 0$.

Remark 4. This corollary recovers and generalizes the results from Section 5.2 in [ArCa04].

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3

Dyadic Elastic Scattering by Point Sources: Direct and Inverse Problems

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3.1 Introduction

This chapter is concerned with the scattering of elastic point sources by a bounded obstacle, as well as with a related near-field inverse problem for small scatterers. We consider the Dirichlet problem, where the displacement field is vanishing on the surface of the scatterer. A dyadic formulation for the aforementioned scattering problem is considered in order to gain the symmetry–compactness of the dyadic analysis [TAI94].

For acoustic and electromagnetic scattering, results on incident waves generated by a point source appear in [DK00], [AMS02]; see also references therein. In all these studies, scattering relations by point sources are established; related simple inversion algorithms for small scatterers can be found in [AMS01]. For elasticity, related problems such as the location and identification of a small three-dimensional elastic inclusion, using arrays of elastic source transmitters and receivers, are considered in [AK04], [ACI08].

This chapter provides results on the direct scattering problem by point-generated elastic waves for the three-dimensional elastic case. Further, a related near-field inversion algorithm for a small rigid sphere in the low-frequency case is established, where the key idea is to measure the scattered field for various point-source locations.

3.2 Governing Equations and Fundamental Solution

In this section, we present the fundamental equations of linearized elasticity and the spectral Navier equation which governs the propagation of time-harmonic waves in an elastic medium. We assume a three-dimensional infinite isotropic and homogeneous elastic medium with Lamé constants λ, μ and mass density ϱ. The Navier equation of the dynamic theory of linearized elasticity is written as [KU65]

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$$\mu\Delta\widetilde{\mathbf{U}}(\mathbf{r},t) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\widetilde{\mathbf{U}}(\mathbf{r},t) = \varrho\,\frac{\partial^2}{\partial t^2}\mathbf{U}(\mathbf{r},\mathbf{t}). \tag{3.1}$$

Assuming the time-spectral decomposition

$$\widetilde{\mathbf{U}}(\mathbf{r},t) = \widetilde{\mathbf{u}}(\mathbf{r})\,e^{-i\omega t}, \tag{3.2}$$

where the circular frequency $\omega > 0$ denotes the Fourier dual variable of t, and the tilde $\sim$ is used to denote dyadic fields, we obtain the spectral (reduced) Navier equation

$$c_s^2\,\Delta\widetilde{\mathbf{u}}(\mathbf{r}) + (c_p^2 - c_s^2)\,\mathrm{grad}\,\mathrm{div}\,\widetilde{\mathbf{u}}(\mathbf{r}) + \omega^2\widetilde{\mathbf{u}}(\mathbf{r}) = \widetilde{\mathbf{0}}, \tag{3.3}$$

where c_p, c_s are the phase velocities of the longitudinal and the transverse wave, respectively, given by

$$c_p = \sqrt{\frac{\lambda+2\mu}{\varrho}}, \quad c_s = \sqrt{\frac{\mu}{\varrho}}. \tag{3.4}$$

An equivalent form of equation (3.3) is given by

$$\mu\,\Delta\widetilde{\mathbf{u}}(\mathbf{r}) + (\lambda+\mu)\;\mathrm{grad}\,\mathrm{div}\,\widetilde{\mathbf{u}}(\mathbf{r}) + \varrho\,\omega^2\widetilde{\mathbf{u}}(\mathbf{r}) = \widetilde{\mathbf{0}}. \tag{3.5}$$

Using the following abbreviation:

$$\Delta^* := \mu\,\Delta + (\lambda+\mu)\;\mathrm{grad}\,\mathrm{div}, \tag{3.6}$$

an alternative form of equation (3.5) (which will be considered from now on) is given by

$$(\Delta^* + \varrho\,\omega^2)\,\widetilde{\mathbf{u}}(\mathbf{r}) = \widetilde{\mathbf{0}}. \tag{3.7}$$

As is well known, under the following assumptions for the Lamé constants:

$$\mu > 0,\;\; \lambda + 2\mu > 0,$$

it can be proved that the Navier equation is uniformly strictly elliptic; hence, the medium sustains both longitudinal and transverse waves.

We note here that any complex-valued solution $\widetilde{\mathbf{u}}$ to the Navier equation (the displacement field) is decomposed as (Helmholtz decomposition)

$$\widetilde{\mathbf{u}}(\mathbf{r}) = \widetilde{\mathbf{u}}^p(\mathbf{r}) + \widetilde{\mathbf{u}}^s(\mathbf{r}), \tag{3.8}$$

where $\widetilde{\mathbf{u}}^p(\mathbf{r})$ is the longitudinal part, while $\widetilde{\mathbf{u}}^s(\mathbf{r})$ is the transverse one. It is well known that $\widetilde{\mathbf{u}}^p(\mathbf{r})$ and $\widetilde{\mathbf{u}}^s(\mathbf{r})$ satisfy the Helmholtz equations

$$(\Delta + k_p^2)\,\widetilde{\mathbf{u}}^p(\mathbf{r}) = \mathbf{0} \quad \text{and} \quad (\Delta + k_s^2)\,\widetilde{\mathbf{u}}^s(\mathbf{r}) = \mathbf{0}, \tag{3.9}$$

respectively. The angular frequency ω is related to the phase velocities c_p and c_s via the relations

$$\omega = k_p \, c_p = c_s k_s, \tag{3.10}$$

where $k_p = 2\pi/\lambda_p$ and $k_s = 2\pi/\lambda_s$ are the wave numbers for the longitudinal and the transverse waves, respectively, and λ_p, λ_s are the corresponding wavelengths.

It is well known that the free-space Green's dyadic of the Navier equation (3.7) is

$$\begin{aligned} \widetilde{\mathbf{\Gamma}}(\mathbf{r}, \mathbf{r}') \quad &= \quad - \frac{ik_p}{4\pi \varrho \omega^2} \, \mathrm{grad}_r \, \mathrm{grad}_r^\top \, \frac{e^{ik_p |\mathbf{r}-\mathbf{r}'|}}{ik_p \, |\mathbf{r}-\mathbf{r}'|} \\ &\quad + \frac{ik_s}{4\pi \varrho \omega^2} \, (\mathrm{grad}_r \, \mathrm{grad}_r^\top + k_s^2 \, \widetilde{\mathbf{I}}) \, \frac{e^{ik_s |\mathbf{r}-\mathbf{r}'|}}{ik_s \, |\mathbf{r}-\mathbf{r}'|}, \end{aligned} \tag{3.11}$$

where "$\top$" denotes transposition, and $\widetilde{\mathbf{I}}$ is the identity dyadic.

3.3 The Direct Scattering Problem

Let B_i be an open bounded region in $\mathbb{R}^3$ with a smooth boundary ∂B. The set B_i will be referred to as the scatterer, while the complement of B_i, which will be denoted by B, is characterized by the Lamé constants λ, μ and density ϱ. In what follows we consider the direct scattering problem for the case of Dirichlet data and C^2-boundary. Other boundary conditions (Neumann, transmission) have been studied in [ASS08], [ASS].

We assume that our scatterer is irradiated by a dyadic incident wave due to a source at a point $\mathbf{a}$, i.e.,

$$\widetilde{\mathbf{u}}_a^{\mathrm{inc}}(\mathbf{r}) = -\frac{ik_p}{\omega^2} \, \mathrm{grad}_r \, \mathrm{grad}_r^\top \, h(k_p \varepsilon) + \frac{ik_s}{\omega^2} \, (\mathrm{grad}_r \, \mathrm{grad}_r^\top + k_s^2 \, \widetilde{\mathbf{I}}) \, h(k_s \varepsilon), \quad \mathbf{r} \neq \mathbf{a}, \tag{3.12}$$

where $\varepsilon := |\mathbf{r} - \mathbf{a}|$ and the function $h(x) := e^{ix}/(ix)$ is the spherical Hankel function of the first kind and zero order. We can prove that when $a = |\mathbf{a}| \to \infty$, the incident point-source field (3.12) reduces to a dyadic plane wave with direction of propagation $-\hat{\mathbf{a}}$, i.e.,

$$\widetilde{\mathbf{u}}^{\mathrm{inc}}(\mathbf{r}; -\hat{\mathbf{a}}) = A_p \, (\hat{\mathbf{a}} \otimes \hat{\mathbf{a}}) \, e^{-ik_p \, \mathbf{r}\cdot\hat{\mathbf{a}}} + A_s (\widetilde{\mathbf{I}} - \hat{\mathbf{a}} \otimes \hat{\mathbf{a}}) \, e^{-ik_s \, \mathbf{r}\cdot\hat{\mathbf{a}}}, \tag{3.13}$$

where A_p, A_s are constants which stand for the corresponding amplitudes, given by

$$A_p := \frac{1}{\lambda + 2\mu} \, \frac{e^{ik_p a}}{a} \quad \text{and} \quad A_s := \frac{1}{\mu} \, \frac{e^{ik_s a}}{a}. \tag{3.14}$$

Note that the first term on the right-hand side of (3.13) describes the incident longitudinal plane wave, while the second one describes the incident transverse plane wave.

We now describe the scattering process, which has to deal with the disturbance that a given obstacle causes upon the propagation of a known wave field. This disturbance for a rigid (Dirichlet boundary condition) scatterer or a cavity (Neumann boundary condition) is expressed by the generation of a scattered dyadic field corresponding to the point-source incident field at **a**; denoted by $\widetilde{\mathbf{u}}_a^{\rm sct}(\mathbf{r})$. Then the total field $\widetilde{\mathbf{u}}_a^{\rm tot}(\mathbf{r})$ in the exterior B of the scatterer is the superposition of the incident and the scattered wave, i.e.,

$$\widetilde{\mathbf{u}}_a^{\rm tot}(\mathbf{r}) = \widetilde{\mathbf{u}}_a^{\rm inc}(\mathbf{r}) + \widetilde{\mathbf{u}}_a^{\rm sct}(\mathbf{r})\,, \quad \mathbf{r} \in B.$$

In addition, the scattered dyadic field $\widetilde{\mathbf{u}}_a^{\rm sct}(\mathbf{r})$ due to the Helmholtz decomposition is written as

$$\widetilde{\mathbf{u}}_a^{\rm sct}(\mathbf{r}) = \widetilde{\mathbf{u}}_a^{\rm sct,p}(\mathbf{r}) + \widetilde{\mathbf{u}}_a^{\rm sct,s}(\mathbf{r}).$$

The differential equation that the aforementioned displacement field satisfies in the region B is given by

$$\varDelta^* \widetilde{\mathbf{u}}_a^{\rm sct}(\mathbf{r}) + \varrho\,\omega^2\,\widetilde{\mathbf{u}}_a^{\rm sct}(\mathbf{r}) = \widetilde{\mathbf{0}}, \quad \mathbf{r} \in B, \tag{3.15}$$

where the differential operator $\varDelta^*$ is defined in (3.6). We introduce now the direct scattering problem which is mathematically described by the following boundary value problem: For a given point-source incident field at **a**, find a solution $\widetilde{\mathbf{u}}_a^{\rm sct} \in C^2(B) \cap C^1(\overline{B})$, such that

$$\varDelta^*\,\widetilde{\mathbf{u}}_a^{\rm sct}(\mathbf{r}) + \varrho\,\omega^2\,\widetilde{\mathbf{u}}_a^{\rm sct}(\mathbf{r}) = \widetilde{\mathbf{0}}, \qquad \mathbf{r} \in B \tag{3.16}$$

$$\widetilde{\mathbf{u}}_a^{\rm sct}(\mathbf{r}) = -\widetilde{\mathbf{u}}_a^{\rm inc}(\mathbf{r}), \qquad \mathbf{r} \in \partial B \tag{3.17}$$

$$\lim_{r\to\infty}\ \widetilde{\mathbf{u}}_a^{{\rm sct},\,\beta} = \tilde{\mathbf{0}}, \qquad r = |\mathbf{r}|, \quad \beta = p, s, \tag{3.18}$$

$$\lim_{r\to\infty} r\,\Big(\frac{\partial \widetilde{\mathbf{u}}_a^{{\rm sct},\,\beta}}{\partial r} - ik_\beta\,\widetilde{\mathbf{u}}_a^{{\rm sct},\,\beta}\Big) = \widetilde{\mathbf{0}}, \qquad r = |\mathbf{r}|, \quad \beta = p, s, \tag{3.19}$$

where relations (3.18)–(3.19) are the well-known radiation conditions which hold uniformly for all directions $r = |\mathbf{r}|$.

We continue the study of our scattering problem, presenting the integral representation for radiating solutions $\widetilde{\mathbf{u}}_a^{\rm sct} \in C^2(B) \cap C^1(\overline{B})$ of the Navier equation (3.7). The latter is obtained (with the use of Betti's formula) and is given by

$$\widetilde{\mathbf{u}}_a^{\rm sct}(\mathbf{r}) = \int_{\partial B} \left[\left(\mathbf{T}^{(r')}\widetilde{\mathbf{\Gamma}}(\mathbf{r},\mathbf{r}')\right)^{\top} \cdot \widetilde{\mathbf{u}}_a^{\rm sct}(\mathbf{r}') - \widetilde{\mathbf{\Gamma}}(\mathbf{r},\mathbf{r}') \cdot \mathbf{T}^{(r')}\widetilde{\mathbf{u}}_a^{\rm sct}(\mathbf{r}')\right] ds(\mathbf{r}'), \tag{3.20}$$

where the superscript denotes the action of the differential operator on the indicated variable, and $\mathbf{T}$ denotes the surface stress operator defined by

$$\mathbf{T} = 2\mu\,\hat{\mathbf{n}}_{\mathbf{r}} \cdot \operatorname{grad} + \lambda\,\hat{\mathbf{n}}_{\mathbf{r}}\operatorname{div} + \mu\,\hat{\mathbf{n}}_{\mathbf{r}} \times \operatorname{curl} \tag{3.21}$$

with $\hat{\mathbf{n}}_{\mathbf{r}}$ being the outward unit normal vector on the C^2-boundary ∂B at the point $\mathbf{r}$, and $\mathbf{r} \in B$. Using now asymptotic analysis for $\widetilde{\mathbf{\Gamma}}(\mathbf{r}, \mathbf{r}')$, the relations for the far-field patterns of the longitudinal and transverse parts, respectively, of the fundamental dyadic $\widetilde{\mathbf{\Gamma}}(\mathbf{r}, \mathbf{r}')$ are given by

$$\widetilde{\mathbf{\Gamma}}^p_\infty(\mathbf{r}, \mathbf{r}') = \frac{ik_p}{\lambda + 2\mu} \; (\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}) \; e^{-ik_p \mathbf{r}' \cdot \hat{\mathbf{r}}}, \qquad r = |\mathbf{r}| \to \infty, \tag{3.22}$$

$$\widetilde{\mathbf{\Gamma}}^s_\infty(\mathbf{r}, \mathbf{r}') = \frac{ik_s}{\mu} \; (\widetilde{\mathbf{I}} - \hat{\mathbf{r}} \otimes \hat{\mathbf{r}}) \, e^{-ik_s \mathbf{r}' \cdot \hat{\mathbf{r}}}, \qquad r = |\mathbf{r}| \to \infty, \tag{3.23}$$

where "$\otimes$" is the juxtaposition between two vectors (this gives a dyadic), and the dyadics $\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}$ and $\widetilde{\mathbf{I}} - \hat{\mathbf{r}} \otimes \hat{\mathbf{r}}$ in (3.22) and (3.23) present the radial and tangential behavior of the longitudinal and transverse parts, respectively, of $\widetilde{\mathbf{\Gamma}}(\mathbf{r}, \mathbf{r}')$ far away from the scatterer at the radiation zone.

With the aid of (3.22)–(3.23) and the integral representation (3.20), any radiating solution has the asymptotic behavior of the form

$$\widetilde{\mathbf{u}}^{\text{sct}}_a(\mathbf{r}) = \widetilde{\mathbf{u}}^{\infty,p}_a(\hat{\mathbf{r}}) \, \frac{e^{ik_p r}}{ik_p r} + \widetilde{\mathbf{u}}^{\infty,s}_a(\hat{\mathbf{r}}) \, \frac{e^{ik_s r}}{ik_s r} + O(r^{-2}), \quad r = |\mathbf{r}| \to \infty, \tag{3.24}$$

uniformly with respect to $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} \in \Omega$, where Ω is the unit sphere. The coefficients of the terms $\frac{e^{ik_\beta r}}{ik_\beta r}$, $\beta = p, s$ are the corresponding dyadic far-field patterns, which are analytic functions defined on the unit sphere Ω in $\mathbb{R}^3$, and are known as the longitudinal and the transverse far-field patterns, respectively.

A comprehensive account of results in linear elasticity can be found in [G72]. Existence and uniqueness of the above direct scattering problem (3.16)–(3.19) have been proved, e.g., in [KU65], [KGBB]. It is well known that, in order to reformulate the direct scattering problem in integral form, we can follow either the direct method, based on Betti's formulae, or the indirect method, using layer potentials; for the use of the boundary integral equations method in the study of a variety of problems, see the recent book [HW08].

The problem of scattering of elastic spherical waves by a rigid body, a cavity, or a penetrable obstacle in three-dimensional linear elasticity has been studied in [ASS08]. In particular, for two point sources, dyadic far-field pattern generators are defined, which are used for the formulation of a general scattering theorem. The main reciprocity principle and mixed scattering relations are also established there.

3.4 A Simple Inversion Algorithm for a Small Sphere

Concerning the three-dimensional case and following the same procedure as in [ASS07], [ASS] for the two-dimensional analogous one, we present the necessary basic formulae connected with the inversion algorithm for the reconstruction of an elastic rigid sphere. We recall that the three-dimensional incident elastic wave due to a point source at $\mathbf{a}$ is

$$\widetilde{\mathbf{u}}_a^{\mathrm{inc}}(\mathbf{r}) = \frac{ik_s}{\omega^2}\,(\mathrm{grad}_r\,\mathrm{grad}_r^\top + k_s^2\,\widetilde{\mathbf{I}})\,h(k_s\varepsilon) - \frac{ik_p}{\omega^2}\,\mathrm{grad}_r\,\mathrm{grad}_r^\top\,h(k_p\varepsilon), \quad (3.25)$$

where $\varepsilon := |\mathbf{r}-\mathbf{a}|,\ \ \mathbf{r} \neq \mathbf{a}$.

Let us now consider the case of a spherical scatterer of radius R. If we take spherical polar coordinates (r,θ,ϕ) and expand the point-source incident field (3.25) in terms of spherical Navier eigenvectors (Hansen vectors) $\mathbf{L}_{mn}^{e,i}, \mathbf{M}_{mn}^{e,i}$, and $\mathbf{N}_{mn}^{e,i}$ [BS81], we have

$$\widetilde{\mathbf{u}}_a^{\mathrm{inc}}(\mathbf{r}) = \frac{ik_s}{\mu}\sum_{n=1,1,0}^{\infty}\sum_{m=0}^{n}\sum_{\sigma=e,o}\frac{1}{\mathfrak{G}_{mn}}\Big[\frac{1}{n(n+1)}\,\overline{\mathbf{M}}_{\sigma mn}^{-}(k_sa)\otimes\mathbf{M}_{\sigma mn}^{+}(k_sr)$$

$$+\frac{1}{n(n+1)}\,\overline{\mathbf{N}}_{\sigma mn}^{-}(k_sa)\otimes\mathbf{N}_{\sigma mn}^{+}(k_sr) + \left(\frac{k_p}{k_s}\right)^3\,\overline{\mathbf{L}}_{\sigma mn}^{-}(k_pa)\otimes\mathbf{L}_{\sigma mn}^{+}(k_pr)\Big],$$

where $r := |\mathbf{r}| < |\mathbf{a}|$, $+(-)$ denotes the interior (exterior) Hansen vector, the overbar stands for a complex conjugate, and

$$\mathfrak{G}_{mn} = \frac{4\pi}{2n+1}\,\frac{(n+m)!}{(n-m)!}.$$

The scattered field has a similar expression and takes the form

$$\widetilde{\mathbf{u}}_a^{\mathrm{sct}}(\mathbf{r}) = \frac{ik_s}{\mu}\sum_{n=1,1,0}^{\infty}\sum_{m=0}^{n}\frac{1}{\mathfrak{G}_{mn}}\Big[\alpha_n^{m,s}\frac{h_n(k_sr)}{\sqrt{n(n+1)}}\,(\overline{\mathbf{M}}_{emn}^{-}(k_sa)\otimes\mathbf{C}_{emn}(\theta,\phi)$$

$$+\overline{\mathbf{M}}_{omn}^{-}(k_sa)\otimes\mathbf{C}_{omn}(\theta,\phi)) + \beta_n^{m,s}\frac{h_n(k_sr)}{k_sr}(\overline{\mathbf{N}}_{emn}^{-}(k_sa)\otimes\mathbf{P}_{emn}(\theta,\phi)$$

$$+\overline{\mathbf{N}}_{omn}^{-}(k_sa)\otimes\mathbf{P}_{omn}(\theta,\phi))$$

$$+\gamma_n^{m,s}\frac{h_n(k_sr)/k_sr + h_n'(k_sr)}{\sqrt{n(n+1)}}(\overline{\mathbf{N}}_{emn}^{-}(k_sa)\otimes\mathbf{B}_{emn}(\theta,\phi)$$

$$+\overline{\mathbf{N}}_{omn}^{-}(k_sa)\otimes\mathbf{B}_{omn}(\theta,\phi))$$

$$+\delta_n^{m,s}\,h_n'(k_pr)\left(\frac{k_p}{k_s}\right)^3\,(\overline{\mathbf{L}}_{emn}^{-}(k_pa)\otimes\mathbf{B}_{emn}(\theta,\phi)$$

$$+\overline{\mathbf{L}}_{omn}^{-}(k_pa)\otimes\mathbf{B}_{omn}(\theta,\phi))$$

$$+\varepsilon_n^{m,s}\sqrt{n(n+1)}\frac{h_n(k_pr)}{k_pr}(\frac{k_p}{k_s})^3(\overline{\mathbf{L}}_{emn}^{-}(k_pa)\otimes\mathbf{B}_{emn}(\theta,\phi)$$

$$+\overline{\mathbf{L}}_{omn}^{-}(k_pa)\otimes\mathbf{B}_{omn}(\theta,\phi)\,)\Big],$$

where the coefficients $\alpha_n^{m,s},\ \beta_n^{m,s},\ \gamma_n^{m,s},\ \delta_n^{m,s}$, and $\varepsilon_n^{m,s}$ are to be determined. The Dirichlet boundary condition (3.17) on $r = R$ (surface of the elastic sphere), and some orthogonality relations, yield

$$\alpha_n^{m,s} = -\frac{j_n(k_s R)}{h_n(k_s R)}, \qquad \beta_n^{m,s} = -\frac{j_n(k_s R)}{h_n(k_s R)},$$

$$\gamma_n^{m,s} = -\frac{j_n(k_s R) + k_s R\, j_n'(k_s R)}{h_n(k_s R) + k_s R\, h_n'(k_s R)},$$

$$\delta_n^{m,s} = -\frac{j_n(k_p R)}{h_n(k_p R)}, \qquad \varepsilon_n^{m,s} = -\frac{j_n(k_p R)}{h_n(k_p R)},$$

where $j_n(k_\beta R),\ \beta = p, s$, are the spherical Bessel functions of first kind and order n.

Finally, a simple inverse near-field method for a small rigid (i.e., Dirichlet boundary condition) sphere can now easily be established. In particular, we can solve the inverse problem using near-field experiments, and following similar steps as for the two-dimensional case [ASS], we can locate the center and the radius of a small rigid sphere. Let us mention here that by the term "small sphere" we mean that we work in the "low-frequency regime," i.e., that $k_\beta R \ll 1,\ \ \beta = p, s$.

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4

Two-Operator Boundary–Domain Integral Equations for a Variable-Coefficient BVP

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4.1 Introduction

Partial differential equations (PDEs) with variable coefficients often arise in mathematical modeling of inhomogeneous media (e.g., functionally graded materials or materials with damage-induced inhomogeneity) in solid mechanics, electromagnetics, heat conduction, fluid flows through porous media, and other areas of physics and engineering.

Generally, explicit fundamental solutions are not available if the PDE coefficients are not constant, preventing formulation of explicit boundary integral equations, which can then be effectively solved numerically. Nevertheless, for a rather wide class of variable-coefficient PDEs, it is possible to use instead an explicit parametrix (Levi function) taken as a fundamental solution of corresponding frozen-coefficient PDEs, and reduce boundary value problems (BVPs) for such PDEs to explicit systems of boundary–domain integral equations (BDIEs); see, e.g., [Mi02, CMN09, Mi06] and references therein. However this (one-operator) approach does not work when the fundamental solution of the frozen-coefficient PDE is not available explicitly (as, e.g., in the Lamé system of anisotropic elasticity).

To overcome this difficulty, one can apply the two-operator approach, formulated in [Mi05] for some nonlinear problems, that employs a parametrix of another (second) PDE, not related with the PDE in question, for reducing the BVP to a BDIE system. Since the second PDE is rather arbitrary, one can always choose it in such a way that its parametrix is available explicitly. The simplest choice for the second PDE is the one with a fundamental solution explicitly available.

To analyze the two-operator approach, we apply in this paper one of its linear versions to the mixed (Dirichlet–Neumann) BVP for a linear second-order scalar elliptic variable-coefficient PDE, reducing it to four different BDIE systems. Although the considered BVP can also be reduced to (other) BDIE systems by the one-operator approach, it can be considered as a simple "toy" model showing the main features of the two-operator approach arising also

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in reducing more general BVPs to BDIEs. The two-operator BDIE systems are nonstandard systems of equations containing integral operators defined on the domain under consideration and potential type and pseudo-differential operators defined on open submanifolds of the boundary. Using the results of [CMN09], we give a rigorous analysis of the two-operator BDIEs and show that the BDIE systems are *equivalent* to the mixed BVP and thus are *uniquely solvable*, while the corresponding boundary–domain integral operators are *invertible* in appropriate Sobolev–Slobodetski (Bessel-potential) spaces.

4.2 Function Spaces and BVP

Let $\Omega = \Omega^+$ be an open three-dimensional region of $\mathbb{R}^3$, $\Omega^-:=\mathbb{R}^3\setminus\overline{\Omega^+}$ and the boundary $\partial\Omega$ be a simply connected, closed, infinitely smooth surface. Moreover, $\partial\Omega = \overline{\partial_D\Omega}\bigcup\overline{\partial_N\Omega}$, where $\partial_D\Omega$ and $\partial_N\Omega$ are open, nonempty, nonintersecting, simply connected submanifolds of $\partial\Omega$ with an infinitely smooth boundary curve $\overline{\partial_D\Omega}\bigcap\overline{\partial_N\Omega} \in C^\infty$. Let $a \in C^\infty(\mathbb{R}^3)$, $a(x) > 0$ and also $\partial_j := \partial/\partial x_j$ $(j = 1, 2, 3)$, $\partial_x = (\partial_1, \partial_2, \partial_3)$. We consider the following PDE with scalar variable coefficient:

$$L_a u(x) := L_a(x, \partial_x)u(x) := \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial u(x)}{\partial x_i}\right] = f(x), \quad x \in \Omega^\pm, \tag{4.1}$$

where u is an unknown function and f is a given function in $\Omega^\pm$.

In what follows, $H^s(\Omega^+) = H_2^s(\Omega^+)$, $H^s_{loc}(\Omega^-) = H^s_{2,\,loc}(\Omega^-)$, $H^s(\partial\Omega) = H_2^s(\partial\Omega)$ denote the Bessel potential spaces (coinciding with the Sobolev–Slobodetski spaces if $s \geq 0$). For $S_1 \subset \partial\Omega$, we will use the subspace $\widetilde{H}^s(S_1) = \{g : \ g \in H^s(\partial\Omega),\ \operatorname{supp}(g) \subset \overline{S_1}\}$ of $H^s(\partial\Omega)$, while $H^s(S_1) = \{r_{S_1} g : \ g \in H^s(\partial\Omega)\}$, where r_{S_1} denotes the restriction operator on S_1.

From the trace theorem (see, e.g., [LiMa72]) for $u \in H^1(\Omega^\pm)$, it follows that $u|^\pm_{\partial\Omega} := \gamma^\pm u \in H^{\frac{1}{2}}(\partial\Omega)$, where $\gamma^\pm$ is the trace operator on $\partial\Omega$ from $\Omega^\pm$. We will use γ for $\gamma^\pm$ if $\gamma^+ = \gamma^-$. We will also use the notation $u^\pm$ for the traces $u|^\pm_{\partial\Omega}$, when this will cause no confusion.

For a linear operator L_* we introduce the following subspace of $H^s(\Omega^\pm)$ [Gr85, Co88]:

$$H^{s,0}(\Omega^\pm; L_*) := \{g \in H^s(\Omega^\pm) : L_* g \in L_2(\Omega^\pm)\},$$

$$\|g\|^2_{H^{s,0}(\Omega^\pm;L_*)} := \|g\|^2_{H^s} + \|L_* g\|^2_{H^0(\Omega^\pm)} = \|g\|^2_{H^s} + \|L_* g\|^2_{L_2(\Omega^\pm)}.$$

In this chapter, we will particularly use the space $H^{1,0}(\Omega^\pm; L_*)$ where L_* is either the operator L_a from (4.1) or the Laplace operator Δ, and one can see that these spaces coincide.

For $u \in H^{1,0}(\Omega^\pm; \Delta)$, we can correctly define the (canonical) co-normal derivative $T_a^\pm u \in H^{-\frac{1}{2}}(\partial\Omega)$, cf. [Co88, McL00, Mi07], as

$$\langle T_a^{\pm} u, w\rangle_{\partial\Omega} := \pm \int_{\Omega^{\pm}} \left[\gamma_{-1}^{\pm} w \cdot L_a u + E_a(u, \gamma_{-1}^{\pm} w)\right] dx \quad \forall\, w \in H^{1/2}(\partial\Omega), \tag{4.2}$$

where $\gamma_{-1}^{\pm} : H^{1/2}(\partial\Omega) \to H^1(\Omega\pm)$ is a right inverse to the trace operator $\gamma^{\pm}$,

$$E_a(u,v) := \sum_{i=1}^{3} a(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i} = a(x)\nabla u(x) \cdot \nabla v(x)$$

and $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the duality brackets between the spaces $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$, which extend the usual $L_2(\partial\Omega)$ inner product; to simplify the notation we will also sometimes write the duality brackets as integral. Then for $u \in H^{1,0}(\Omega^{\pm};\Delta)$, $v \in H^1(\Omega)$, the first Green identity holds [Co88, Lemma 3.4], [Mi07, Lemma 4.8],

$$\int_{\Omega^{\pm}} v(x) L_a u(x) dx = \pm \int_{\partial\Omega} v(x) T_a^{+} u(x) dS(x) - \int_{\Omega^{\pm}} E_a(u,v) dx \, . \tag{4.3}$$

If $u \in H^2(\Omega^{\pm})$, the canonical co-normal derivative $T_a^{\pm} u$ defined by (4.2) reduces to its classical form

$$T_a^{\pm} u := \sum_{i=1}^{3} a(x) n_i(x) \left[\frac{\partial u(x)}{\partial x_i}\right]^{\pm} = a(x) \left[\frac{\partial u(x)}{\partial n(x)}\right]^{\pm}, \tag{4.4}$$

where $n(x)$ is the exterior (to $\Omega^{\pm}$) unit normal at the point $x \in \partial\Omega$.

We will derive and investigate the *two-operator boundary–domain integral equation systems* for the following mixed boundary value problem:

$$L_a u = f \quad \text{in } \Omega \tag{4.5}$$

$$u^{+} = \varphi_0 \quad \text{on } \partial_D\Omega \tag{4.6}$$

$$T_a^{+} u = \psi_0 \quad \text{on } \partial_N\Omega, \tag{4.7}$$

where $\varphi_0 \in H^{\frac{1}{2}}(\partial_D\Omega)$, $\psi_0 \in H^{-\frac{1}{2}}(\partial_N\Omega)$, and $f \in L_2(\Omega)$. Equation (4.5) is understood in the distributional sense, condition (4.6) in the trace sense, and equality (4.7) in the functional sense (4.2).

Let us consider another auxiliary linear elliptic partial differential operator L_b such that

$$L_b u(x) := L_b(x, \partial_x) u(x) := \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left[b(x) \frac{\partial u(x)}{\partial x_i}\right], \tag{4.8}$$

where $b \in C^{\infty}(\mathbb{R}^3)$, $b(x) > 0$. Then for $u \in H^{1,0}(\Omega^{\pm};\Delta) = H^{1,0}(\Omega^{\pm};\Delta)$ the associate co-normal derivative operator $T_b^{\pm}$ is defined by (4.2) (and for $u \in H^2(\Omega^{\pm})$ by (4.4)) with a replaced by b. If $v \in H^{1,0}(\Omega^{\pm};\Delta)$, $u \in H^1(\Omega)$, then for the operator L_b the first Green identity holds:

$$\int_{\Omega^\pm} u(x)L_b v(x)dx = \pm\int_{\partial\Omega} u(x)T_b^\pm v(x)dS - \int_{\Omega^\pm} E_b(u,v)dx. \tag{4.9}$$

If $u, v \in H^{1,0}(\Omega^\pm;\Delta)$, then subtracting (4.3) from (4.9), we obtain the *two-operator second Green identity*, cf. [Mi05],

$$\int_{\Omega^\pm} \{u(x)L_b v(x) - v(x)L_a u(x)\}\,dx = \pm\int_{\partial\Omega} \left\{u(x)T_b^+ v(x) - v(x)T_a^+ u(x)\right\} dS + \int_{\Omega^\pm} [a(x)-b(x)]\nabla v(x)\cdot\nabla u(x)dx \tag{4.10}$$

Note that if $a = b$, then the last domain integral disappears, and the two-operator Green identity degenerates into the classical second Green identity.

4.3 Parametrix and Potential-Type Operators

As follows from [Mir70, Mi02, CMN09], the function

$$P_b(x,y) = \frac{-1}{4\pi b(y)|x-y|}, \qquad x, y \in \mathbb{R}^3 \tag{4.11}$$

is a parametrix (Levi function) for the operator $L_b(x;\partial_x)$ from (4.8), i.e., it satisfies the equation

$$L_b(x,\partial_x)P_b(x,y) = \delta(x-y) + R_b(x,y)$$

with

$$R_b(x,y) = \sum_{i=1}^{3} \frac{x_i - y_i}{4\pi b(y)|x-y|^3}\frac{\partial b(x)}{\partial x_i}, \qquad x, y \in \mathbb{R}^3. \tag{4.12}$$

Evidently, the parametrix $P_b(x,y)$ is a fundamental solution to the operator $L_b(y,\partial_x) := b(y)\Delta(\partial_x)$ with “frozen” coefficient $b(x) = b(y)$, i.e.,

$$L_b(y,\partial_x)P_b(x,y) = \delta(x-y).$$

The parametrix-based Newtonian and the remainder volume potential operators, corresponding to the parametrix (4.11) and to remainder (4.12), are given, respectively, by

$$\mathcal{P}_b g(y) := \int_\Omega P_b(x,y)g(x)dx, \quad \mathcal{R}_b g(y) := \int_\Omega R_b(x,y)g(x)dx. \tag{4.13}$$

Let us introduce the single-layer and the double-layer surface potential operators, based on parametrix (4.11),

$$V_b g(y) := -\int_{\partial\Omega} P_b(x,y) g(x) dS_x, \qquad y \notin \partial\Omega, \tag{4.14}$$

$$W_b g(y) := -\int_{\partial\Omega} [T_b(x,n(x),\partial_x) P_b(x,y)] g(x) dS_x, \qquad y \notin \partial\Omega. \tag{4.15}$$

For $y \in \partial\Omega$, the corresponding boundary integral (pseudo-differential) operators of direct surface values of the single-layer potential $\mathcal{V}_b$ and the double-layer potential $\mathcal{W}_b$ are

$$\mathcal{V}_b g(y) := -\int_{\partial\Omega} P_b(x,y) g(x) dS_x, \tag{4.16}$$

$$\mathcal{W}_b g(y) := -\int_{\partial\Omega} [T_b(x,n(x),\partial_x) P_b(x,y)] g(x) dS_x. \tag{4.17}$$

We can also calculate at $y \in \partial\Omega$ the co-normal derivatives, associated with the operator L_a, of the single-layer potential and of the double-layer potential,

$$T_a^{\pm} V_b g(y) = \frac{a(y)}{b(y)} T_b^{\pm} V_b g(y), \tag{4.18}$$

$$\mathcal{L}_{ab}^{\pm} g(y) := T_a^{\pm} W_b g(y) = \frac{a(y)}{b(y)} T_b^{\pm} W_b g(y) =: \frac{a(y)}{b(y)} \mathcal{L}_b^{\pm} g(y). \tag{4.19}$$

The direct value operators associated with (4.18) are

$$\mathcal{W}'_{ab} g(y) := -\int_{\partial\Omega} [T_a(y,n(y),\partial_y) P_b(x,y)] g(x) dS_x = \frac{a(y)}{b(y)} \mathcal{W}'_b g(y), \tag{4.20}$$

$$\mathcal{W}'_b g(y) := -\int_{\partial\Omega} [T_b(y,n(y),\partial_y) P_b(x,y)] g(x) dS_x. \tag{4.21}$$

From equations (4.13)–(4.21) we deduce representations of the parametrix-based surface potential boundary operators in terms of their counterparts for $b = 1$, that is, associated with the fundamental solution $P_\Delta = -(4\pi|x-y|)^{-1}$ of the Laplace operator Δ.

$$\mathcal{P}_b g = \frac{1}{b}\mathcal{P}_\Delta g, \quad \mathcal{R}_b g = -\frac{1}{b}\sum_{j=1}^{3} \partial_j \mathcal{P}_\Delta \left[g(\partial_j b)\right], \tag{4.22}$$

$$\frac{a}{b} V_a g = V_b g = \frac{1}{b} V_\Delta g; \qquad \frac{a}{b} W_a \left(\frac{bg}{a}\right) = W_b g = \frac{1}{b} W_\Delta (bg), \tag{4.23}$$

$$\frac{a}{b} \mathcal{V}_a g = \mathcal{V}_b g = \frac{1}{b} \mathcal{V}_\Delta g; \qquad \frac{a}{b} \mathcal{W}_a \left(\frac{bg}{a}\right) = \mathcal{W}_b g = \frac{1}{b} \mathcal{W}_\Delta (bg), \tag{4.24}$$

$$\mathcal{W}'{}_{ab} g = \frac{a}{b} \mathcal{W}'{}_b g = \frac{a}{b} \left\{ \mathcal{W}'{}_\Delta (bg) + \left[b \frac{\partial}{\partial n} \left(\frac{1}{b}\right) \right] \mathcal{V}_\Delta g \right\}, \tag{4.25}$$

$$\mathcal{L}_{ab}^{\pm} g := \frac{a}{b} \mathcal{L}_b^{\pm} g = \frac{a}{b} \left\{ \mathcal{L}_\Delta (bg) + \left[b \frac{\partial}{\partial n} \left(\frac{1}{b}\right) \right] W_\Delta^{\pm} (bg) \right\}. \tag{4.26}$$

It is taken into account that b and its derivatives are continuous in $\mathbb{R}^3$ and $\mathcal{L}_\Delta(bg) := \mathcal{L}^+_\Delta(bg) = \mathcal{L}^-_\Delta(bg)$ by the Liapunov–Tauber theorem.

The mapping properties of the volume and surface potentials are proved in [CMN09], see also Appendices A and B in [Mi06]. Similar to Theorems 3.3 and 3.6 in [CMN09] (see also Appendices A and B in [Mi06]), relations (4.23)–(4.26) imply the two following jump relation theorems.

Theorem 1. *Let $g_1 \in H^{-\frac{1}{2}}(\partial\Omega)$, and $g_2 \in H^{\frac{1}{2}}(\partial\Omega)$. Then the following relations hold on $\partial\Omega$:*

$$\begin{aligned}
[V_b g_1]^{\pm} &= \mathcal{V}_b g_1,\\
[W_b g_2]^{\pm} &= \mp\frac{1}{2} g_2 + \mathcal{W}_b g_2,\\
T_a^{\pm} V_b g_1 &= \pm\frac{1}{2}\frac{a}{b} g_1 + \mathcal{W}'_{ab} g_1.
\end{aligned}$$

Theorem 2. *Let S_1 and $\partial\Omega\setminus\overline{S}_1$ be nonempty, open, simply connected submanifolds of $\partial\Omega$ with an infinitely smooth boundary curve, and $0 < s < 1$. Then*

$$\mathcal{L}^+_{ab} + \frac{a}{b}\frac{\partial b}{\partial n}\left(-\frac{1}{2}I + \mathcal{W}_b\right) = \mathcal{L}^-_{ab} + \frac{a}{b}\frac{\partial b}{\partial n}\left(\frac{1}{2}I + \mathcal{W}_b\right) \quad \text{on } \partial\Omega.$$

Moreover, the pseudo-differential operator $r_{S_1}\widehat{\mathcal{L}}_{ab} : \widetilde{H}^s(S_1) \to H^{s-1}(S_1)$, where

$$\widehat{\mathcal{L}}_{ab} g := \left[\frac{b}{a}\mathcal{L}^{\pm}_{ab} + \frac{\partial b}{\partial n}\left(\mp\frac{1}{2}I + \mathcal{W}_b\right)\right] g = \mathcal{L}_\Delta(bg) \quad \text{on } \partial\Omega,$$

is invertible, while the operators $r_{S_1}\left(\frac{b}{a}\mathcal{L}^{\pm}_{ab} - \widehat{\mathcal{L}}_{ab}\right) : \widetilde{H}^s(S_1) \to H^s(S_1)$ are bounded and the operators $r_{S_1}\left(\frac{b}{a}\mathcal{L}^{\pm}_{ab} - \widehat{\mathcal{L}}_{ab}\right) : \widetilde{H}^s(S_1) \to H^{s-1}(S_1)$ are compact.

For $v(x) := P_b(x, y)$ and $u \in H^{1,0}(\Omega; \Delta)$, we obtain from (4.10) by standard limiting procedures (cf. [Mir70]) the *two-operator third Green identity*,

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b T_a^+ u + W_b u^+ = \mathcal{P}_b L_a u \quad \text{in } \Omega, \tag{4.27}$$

where

$$\begin{aligned}
\mathcal{Z}_b u(y) &:= -\int_\Omega [a(x) - b(x)]\nabla_x P_b(x, y)\cdot\nabla u(x)dx\\
&= \frac{1}{b(y)}\sum_{j=1}^{3}\partial_j\mathcal{P}_\Delta\left[(a-b)\partial_j u\right](y), \quad y \in \Omega.
\end{aligned} \tag{4.28}$$

Using the Gauss divergence theorem, we can rewrite $\mathcal{Z}_b u(y)$ in the form that does not involve derivatives of u,

$$\mathcal{Z}_b u(y) = \left[\frac{a(y)}{b(y)} - 1\right] u(y) + \widehat{\mathcal{Z}}_b u(y), \tag{4.29}$$

$$\widehat{\mathcal{Z}}_b u(y) := \frac{a(y)}{b(y)} W_a u^+(y) - W_b u^+(y) + \frac{a(y)}{b(y)} \mathcal{R}_a u(y) - \mathcal{R}_b u(y), \tag{4.30}$$

which allows us to call $\mathcal{Z}_b$ an integral operator in spite of its integro-differential ansatz (4.28).

Note that substituting (4.29)–(4.30) to (4.27) and multiplying by $b(y)/a(y)$ one reduces (4.27) to the one-operator parametrix-based third Green identity obtained in [CMN09],

$$u + \mathcal{R}_a u - V_a T_a^+ u + W_a u^+ = \mathcal{P}_a L_a u \qquad \text{in} \ \ \Omega.$$

Relations (4.28)–(4.30) and the mapping properties of $\mathcal{P}_\Delta$, $\mathcal{R}_a$, $\mathcal{R}_b$, W_a, and W_b, given by Theorems 3.1, 3.8 in [CMN09], imply the following statement.

Theorem 3. *The operators*

$$\mathcal{Z}_b \ : \ H^s(\Omega) \to H^s(\Omega), \quad s > \frac{1}{2},$$
$$\widehat{\mathcal{Z}}_b \ : \ H^s(\Omega) \to H^{s,0}(\Omega; \Delta), \quad s \geq 1,$$

are continuous.

If $u \in H^{1,0}(\Omega; \Delta)$ is a solution of equation (4.5) with $f \in L_2(\Omega)$, then (4.27) gives

$$Gu := u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b T_a^+ u + W_b u^+ = \mathcal{P}_b f \qquad \text{in} \quad \Omega, \tag{4.31}$$

$$\mathcal{G}u := \frac{1}{2} u^+ + \mathcal{Z}_b^+ u + \mathcal{R}_b^+ u - \mathcal{V}_b T_a^+ u + \mathcal{W}_b u^+ = [\mathcal{P}_b f]^+ \qquad \text{on} \quad \partial\Omega, \tag{4.32}$$

$$\mathcal{T}u := \left(1 - \frac{a}{2b}\right) T_a^+ u + T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u - \mathcal{W}'_{ab} T_a^+ u + \mathcal{L}_{ab}^+ u^+ = T_a^+ \mathcal{P}_b f \qquad \text{on} \quad \partial\Omega, \tag{4.33}$$

where $\mathcal{Z}_b^+ u = [\mathcal{Z}_b u]^+$ and $\mathcal{R}_b^+ u = [\mathcal{R}_b u]^+$.

Note that if $\mathcal{P}_b$ is not only the parametrix but also a fundamental solution of the operator L_b, then the remainder operator $\mathcal{R}_b$ vanishes in (4.31)–(4.33) (and everywhere in the paper), while the operator $\mathcal{Z}_b$ stays unless $L_a = L_b$.

For some functions f, Ψ, Φ, let us consider a more general "indirect" integral relation, associated with (4.31),

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b \Psi + W_b \Phi = \mathcal{P}_b f, \qquad \text{in} \ \ \Omega. \tag{4.34}$$

Similar to the proof of Lemma 4.1 in [CMN09], one can prove the following.

Lemma 1. *Let* $f \in L_2(\Omega)$, $\Psi \in H^{-\frac{1}{2}}(\partial\Omega)$, $\Phi \in H^{\frac{1}{2}}(\partial\Omega)$, *and* $u \in H^1(\Omega)$ *satisfy (4.34). Then* $u \in H^{1,0}(\Omega;\Delta)$, $L_a u = f$ *in* Ω, *and*

$$V_b\left(\Psi - T_a^+ u\right) - W_b\left(\Phi - u^+\right) = 0 \qquad \text{in } \Omega.$$

4.4 Two-Operator Boundary–Domain Integral Equations

Let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some extensions of the given data $\varphi_0 \in H^{\frac{1}{2}}(\partial_D\Omega)$ from $\partial_D\Omega$ to $\partial\Omega$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial_N\Omega)$ from $\partial_N\Omega$ to $\partial\Omega$, respectively. Let us also denote

$$F_0 := \mathcal{P}_b f + V_b\Psi_0 - W_b\Phi_0 \qquad \text{in} \qquad \Omega.$$

Note that for $f \in L_2(\Omega)$, $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$, and $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$, we have the inclusion $F_0 \in H^{1,0}(\Omega, L_a)$ due to the mapping properties of the Newtonian (volume) and layer potentials (cf. Theorems 3.1 and 3.10 in [CMN09]).

To reduce BVP (4.5)–(4.7) to one or another two-operator BDIE system, we will use equation (4.31) in Ω, and restrictions of equation (4.32) or (4.33) on appropriate parts of the boundary. We will always substitute $\Phi_0+\varphi$ for u^+ and $\Psi_0+\psi$ for T_a^+u, cf. [CMN09], where $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ are considered as known, while ψ belongs to $\widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega)$ and φ to $\widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ due to the boundary conditions (4.6)–(4.7) and are to be found along with $u \in H^{1,0}(\Omega;\Delta)$. This will lead us to *segregated* BDIE systems.

4.4.1 The Integral Equation System ($\mathcal{GT}$)

Let us use equation (4.31) in Ω, the restriction of equation (4.32) on $\partial_D\Omega$, and the restriction of equation (4.33) on $\partial_N\Omega$. Then we arrive at the following two-operator segregated system of BDIEs:

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b\psi + W_b\varphi = F_0 \qquad \text{in} \quad \Omega, \tag{4.35}$$
$$\mathcal{Z}_b^+ u + \mathcal{R}_b^+ u - \mathcal{V}_b\psi + \mathcal{W}_b\varphi = F_0^+ - \varphi_0 \qquad \text{on} \quad \partial_D\Omega, \tag{4.36}$$
$$T_a^+\mathcal{Z}_b u + T_a^+\mathcal{R}_b u - \mathcal{W}'_{ab}\psi + \mathcal{L}_{ab}^+\varphi = T_a^+F_0 - \psi_0 \qquad \text{on} \quad \partial_N\Omega \tag{4.37}$$

Note that due to Lemma 1, all terms of equation (4.35) belong to $H^{1,0}(\Omega;\Delta)$ and their co-normal derivatives are well defined. System (4.35)–(4.37) can be rewritten in the form

$$\mathcal{A}^{\mathcal{GT}}\mathcal{U} = \mathcal{F}^{\mathcal{GT}},$$

where

$$
\begin{aligned}
\mathcal{U}^\top &:= [u,\psi,\varphi] \in H^1(\Omega)\times \widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega),\\
\mathcal{F}^{\mathcal{G}\mathcal{T}} &:= [F_0, r_{\partial_D\Omega}F_0^+ - \varphi_0,\ r_{\partial_N\Omega}T_a^+F_0 - \psi_0]^\top,\\
\mathcal{A}^{\mathcal{G}\mathcal{T}} &:= \begin{bmatrix} I+\mathcal{Z}_b+\mathcal{R}_b & -V_b & W_b\\ r_{\partial_D\Omega}[\mathcal{Z}_b^+ + \mathcal{R}_b^+] & -r_{\partial_D\Omega}\mathcal{V}_b & r_{\partial_D\Omega}\mathcal{W}_b\\ r_{\partial_N\Omega}T_a^+[\mathcal{Z}_b+\mathcal{R}_b] & -r_{\partial_N\Omega}\mathcal{W}'_{ab} & r_{\partial_N\Omega}\mathcal{L}_{ab}^+ \end{bmatrix}.
\end{aligned}
$$

4.4.2 The Integral Equation System ($\mathcal{G}\mathcal{G}$)

To obtain another system, we will use equation (4.31) in Ω and equation (4.32), associated with the operator $\mathcal{G}$ on the whole boundary $\partial\Omega$, and arrive at the two-operator segregated BDIE system ($\mathcal{G}\mathcal{G}$),

$$u+\mathcal{Z}_bu+\mathcal{R}_bu - V_b\psi + W_b\varphi = F_0 \quad \text{in} \quad \Omega, \tag{4.38}$$

$$\frac{1}{2}\varphi + \mathcal{Z}_b^+u + \mathcal{R}_b^+u - \mathcal{V}_b\psi + \mathcal{W}_b\varphi = F_0^+ - \Phi_0 \quad \text{on} \quad \partial\Omega\,. \tag{4.39}$$

System (4.38)–(4.39) can be written in the form

$$\mathcal{A}^{\mathcal{G}\mathcal{G}}\mathcal{U} = \mathcal{F}^{\mathcal{G}\mathcal{G}},$$

where

$$
\begin{aligned}
\mathcal{F}^{\mathcal{G}\mathcal{G}} &:= [F_0,\ F_0^+ - \Phi_0]^\top,\\
\mathcal{U}^\top &:= [u,\psi,\varphi] \in H^1(\Omega)\times \widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega),\\
\mathcal{A}^{\mathcal{G}\mathcal{G}} &:= \begin{bmatrix} I+\mathcal{Z}_b+\mathcal{R}_b & -V_b & W_b\\ \mathcal{Z}_b^+ + \mathcal{R}_b^+ & -\mathcal{V}_b & \frac{1}{2}I+\mathcal{W}_b \end{bmatrix}.
\end{aligned}
$$

4.4.3 The Integral Equation System ($\mathcal{T}\mathcal{T}$)

To obtain one more system, we will use equation (4.31) in Ω and equation (4.33) on $\partial\Omega$ and arrive at the two-operator segregated BDIE system ($\mathcal{T}\mathcal{T}$),

$$u+\mathcal{Z}_bu+\mathcal{R}_bu - V_b\psi + W_b\varphi = F_0 \quad \text{in } \Omega, \tag{4.40}$$

$$\left(1-\frac{a}{2b}\right)\psi + T_a^+\mathcal{Z}_bu + T_a^+\mathcal{R}_bu - \mathcal{W}'_{ab}\psi + \mathcal{L}_{ab}^+\varphi = T_a^+F_0 - \Psi_0 \quad \text{on } \partial\Omega. \tag{4.41}$$

System (4.40)–(4.41) can be written in the form

$$\mathcal{A}^{\mathcal{T}\mathcal{T}}\mathcal{U} = \mathcal{F}^{\mathcal{T}\mathcal{T}},$$

where

$$\mathcal{F}^{\mathcal{TT}} := [F_0,\ T_a^+F_0^+ - \Psi_0]^\top,$$
$$\mathcal{U}^\top := [u,\psi,\varphi] \in H^1(\Omega)\times \widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega),$$
$$\mathcal{A}^{\mathcal{TT}} := \begin{bmatrix} I+\mathcal{Z}_b+\mathcal{R}_b & -V_b & W_b \\ T_a^+[\mathcal{Z}_b+\mathcal{R}_b] & (1-\frac{a}{2b})I-\mathcal{W}'_{ab} & \mathcal{L}^+_{ab} \end{bmatrix}.$$

4.4.4 The Integral Equation System ($\mathcal{TG}$)

To reduce BVP (4.5)–(4.7) to a BDIE system of "almost" the second kind (up to the spaces), we will use equation (4.31) in Ω, the restriction of equation (4.33) on $\partial_D\Omega$, and the restriction of equation (4.32) on $\partial_N\Omega$. Then we arrive at the following two-operator segregated BDIE system ($\mathcal{TG}$):

$$u+\mathcal{Z}_b u+\mathcal{R}_b u - V_b\psi + W_b\varphi = F_0 \quad \text{in } \Omega, \tag{4.42}$$
$$\left(1-\frac{a}{2b}\right)T_a^+\mathcal{Z}_b u + T_a^+\mathcal{R}_b u - \mathcal{W}'_{ab}\psi + \mathcal{L}^+_{ab}\varphi = T_a^+F_0 - \Psi_0 \quad \text{on } \partial_D\Omega, \tag{4.43}$$
$$\frac{1}{2}\varphi + \mathcal{Z}_b^+ u + \mathcal{R}_b^+ u - \mathcal{V}_a\psi + \mathcal{W}_a\varphi = F_0^+ - \Phi_0 \quad \text{on } \partial_N\Omega. \tag{4.44}$$

System (4.42)–(4.44) can be rewritten in the form

$$\mathcal{A}^{\mathcal{TG}}\mathcal{U} = \mathcal{F}^{\mathcal{TG}},$$

where

$$\mathcal{F}^{\mathcal{TG}} := [F_0,\ r_{\partial_D\Omega}(T_a^+F_0-\Psi_0),\ r_{\partial_N\Omega}(F_0^+-\Phi_0)]^\top,$$
$$\mathcal{U}^\top := [u,\psi,\varphi] \in H^1(\Omega)\times \widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega),$$
$$\mathcal{A}^{\mathcal{TG}} := \begin{bmatrix} I+\mathcal{Z}_b+\mathcal{R}_b & -V_b & W_b \\ r_{\partial_D\Omega}T_a^+[\mathcal{Z}_b+\mathcal{R}_b] & (1-\frac{a}{2b})I - r_{\partial_D\Omega}\mathcal{W}'_{ab} & r_{\partial_D\Omega}\mathcal{L}^+_{ab} \\ r_{\partial_N\Omega}[\mathcal{Z}_b^+ + \mathcal{R}_b^+] & -r_{\partial_N\Omega}\mathcal{V}_b & \frac{1}{2}I + r_{\partial_N\Omega}\mathcal{W}_b \end{bmatrix}.$$

4.4.5 Equivalence and Invertibility

Using the arguments similar to the proofs of Theorems 5.2, 5.6, 5.9, and 5.12 in [CMN09], one can prove the following equivalence theorem.

Theorem 4. *Let $f \in L_2(\Omega)$ and let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some fixed extensions of $\varphi_0 \in H^{\frac{1}{2}}(\partial_D\Omega)$ and $\psi_0 \in H^{-\frac{1}{2}}$, respectively.*

(i) If some $u \in H^1(\Omega)$ solves the mixed BVP (4.5)–(4.7) in Ω, then the solution is unique and the triple $(u,\psi,\varphi) \in H^1(\Omega)\times\widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times\widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$, where

$$\psi = T_a^+u - \Psi_0, \qquad \varphi = u^+ - \Phi_0 \qquad \text{on} \quad \partial\Omega, \tag{4.45}$$

solves BDIE systems ($\mathcal{GT}$), ($\mathcal{GG}$), ($\mathcal{TT}$), and ($\mathcal{TG}$).

(ii) *Vice versa, if a triple* $(u,\psi,\varphi)\in H^1(\Omega)\times\widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times\widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ *solves BDIE system* $(\mathcal{GT})$ *or* $(\mathcal{GG})$ *or* $(\mathcal{TT})$ *or* $(\mathcal{TG})$*, then the solution is unique, u solves BVP (4.5)–(4.7), and relations (4.45) hold.*

Application of the representation Lemma 5.13 and Corollary 5.14 as well as Corollary 5.16 about invertibility of the mixed BVP (4.5)–(4.7) operator, from [CMN09], along with the equivalence Theorem 4 above, lead to the following invertibility result.

Theorem 5. *The following operators are continuously invertible:*

$$\mathcal{A}^{\mathcal{GG}}\ :\ H^{1,0}(\Omega;\Delta)\times\widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times\widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)\to H^{1,0}(\Omega;\Delta)\times H^{\frac{1}{2}}(\partial\Omega),$$
$$\mathcal{A}^{\mathcal{TT}}\ :\ H^{1,0}(\Omega;\Delta)\times\widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times\widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)\to H^{1,0}(\Omega;\Delta)\times H^{-\frac{1}{2}}(\partial\Omega),$$
$$\mathcal{A}^{\mathcal{GT}}\ :\ H^{1,0}(\Omega;\Delta)\times\widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times\widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)\to H^{1,0}(\Omega;\Delta)\times H^{\frac{1}{2}}(\partial_D\Omega)\times H^{-\frac{1}{2}}(\partial_N\Omega),$$
$$\mathcal{A}^{\mathcal{TG}}\ :\ H^{1,0}(\Omega;\Delta)\times\widetilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times\widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)\to H^{1,0}(\Omega;\Delta)\times H^{-\frac{1}{2}}(\partial_D\Omega)\times H^{\frac{1}{2}}(\partial_N\Omega).$$

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5

Solution of a Class of Nonlinear Matrix Differential Equations with Application to General Relativity

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5.1 Introduction

Five-dimensional general relativity (5DGR) or Kaluza–Klein theory (KKT) [Le84] is considered as a first step towards unification of electromagnetism and gravitation. 5DGR action may be extended by appropriate quadratic terms making up the Gauss–Bonnet term (GBT) to obtain more generalized field equations including up to second-order derivatives of the metric [Lo71]. If cylindrical symmetry is assumed, the 5-metric takes the form [AzCl96, Az08]

$$\mathrm{d}s^2 = -\,\mathrm{d}\rho^2 + \lambda_{ab}(\rho)\,\mathrm{d}x^a\,\mathrm{d}x^b\,, \tag{5.1}$$

where $a, b = 2, \ldots, 5$ and $\lambda_{ab}(\rho)$ is a $4{\times}4$ real symmetric matrix of signature $(-\,-\,+\,-)$. This 5-metric possesses four commuting Killing vectors ${\xi_a}^A \equiv {\delta_a}^A$ $(A = 1, \ldots, 5)$ and is written in the associated coordinates: ($x^1 = \rho$, $x^2 = \varphi$, $x^3 = z$, $x^4 = t$, x^5) where x^2 and x^5 are periodic, x^4 is timelike, and ρ is a radial coordinate. Let "$,\rho$" denote the derivative "$\mathrm{d}/\mathrm{d}\rho$," the field equations describing stationary cylindrically symmetric 5-spacetimes split into a nonlinear $4{\times}4$ matrix differential equation (5.2) and a scalar one (5.3),

$$\begin{aligned}&2\chi_{,\rho} + 4\mathrm{tr}\,\chi_{,\rho} + (\mathrm{tr}\,\chi)\chi + \mathrm{tr}\,\chi^2 + (\mathrm{tr}\,\chi)^2 + \gamma\left\{(\chi^3)_{,\rho} - (\mathrm{tr}\,\chi)(\chi^2)_{,\rho}\right.\\ &+ [(\mathrm{tr}\,\chi)^2 - \mathrm{tr}\,\chi^2]\chi_{,\rho} - (\mathrm{tr}\,\chi_{,\rho})[\chi^2 - (\mathrm{tr}\,\chi)\chi] - (1/2)(\mathrm{tr}\,\chi^2)_{,\rho}\chi\\ &\left.+ (1/2)[(\mathrm{tr}\,\chi)\chi^3 - (\mathrm{tr}\,\chi^2)\chi^2 - (\mathrm{tr}\,\chi)(\mathrm{tr}\,\chi^2)\chi + (\mathrm{tr}\,\chi)^3\chi]\right\} = 0\,,\end{aligned} \tag{5.2}$$

$$\begin{aligned}&6\mathrm{tr}\,B + (\mathrm{tr}\,\chi)^2 - \mathrm{tr}\,\chi^2 + \gamma\left\{\mathrm{tr}\,(B\chi^2) - \mathrm{tr}\,(B\,\chi)\mathrm{tr}\,\chi\right.\\ &\left.+ (1/2)\mathrm{tr}\,B[(\mathrm{tr}\,\chi)^2 - \mathrm{tr}\,\chi^2]\right\} = 0\,,\end{aligned} \tag{5.3}$$

where $\chi(\rho) \equiv \lambda^{-1}\lambda_{,\rho}$ and $B(\rho) \equiv \chi_{,\rho} + (1/2)\chi^2$.

Some 4-stationary solutions to equations (5.2) and (5.3) have been constructed either analytically or by a perturbation approach and interpreted as neutral, charged, or superconducting cosmic strings [AzCl96]. Very recently, the superconducting cosmic string has been reconsidered and generalized [Az08].

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The purpose of this contribution is to prove some properties of equations (5.2) and (5.3) and use them to investigate the set of all their 4-stationary solutions. In Section 5.2, we reduce equations (5.2) and (5.3) and show that any solution χ commutes with its derivatives $\chi_{,\rho}$. Exploiting the latter property, we prove that any solution χ is a polynomial in a constant matrix with scalar coefficients (5.18). In Section 5.3, we discuss singular solutions ($\det\chi \equiv 0$) and in Section 5.4 we deal with regular solutions ($\det\chi \neq 0$). We will either construct rigorously exact solutions or show the nonexistence of solutions.

5.2 Symmetries and Properties

Equation (5.2) is readily brought to the form

$$2Q_{,\rho} + fQ + 2G + \gamma G_{,\rho}\chi - \gamma G\chi^2 = 0\,,$$

where $Q(\rho)$ is a 4×4 real matrix defined by

$$Q \stackrel{\text{def}}{=} 4f + (2 - \gamma G)\chi - \gamma f\chi^2 + \gamma\chi^3\,. \tag{5.4}$$

The invariants of χ are the functions $f(\rho) \equiv \operatorname{tr}\chi$, $g(\rho) \equiv \operatorname{tr}\chi^2$, $h(\rho) \equiv \operatorname{tr}\chi^3$, $k(\rho) \equiv \det\chi$, and $G(\rho) \equiv g - f^2$ and $H(\rho) \equiv h - f^3$. Combining (5.3) and the trace of (5.2) and using the Cayley–Hamilton equation

$$\chi^4 = f\chi^3 + (G/2)\chi^2 + [(H/3) - (fG/2)]\chi - k \tag{5.5}$$

to eliminate $\operatorname{tr}\chi^4$ leads to (5.7). Hence, any solution to the system (5.2 and 5.3) is necessarily a solution to the following reduced system (5.6 and 5.7):

$$2Q_{,\rho} + fQ + 2G + \gamma G_{,\rho}\chi - \gamma G\chi^2 = 0\,, \tag{5.6}$$

$$24\gamma k = G(8 + 2\gamma G + 3\gamma f^2 + 2\gamma f_{,\rho})\,. \tag{5.7}$$

The system (5.2 and 5.3) or its reduced form (5.6 and 5.7) remains invariant if one performs a linear coordinate transformation with constant coefficients mixing the four Killing vectors together and their associated cyclic coordinates

$$x^a = S^a{}_b\, x^b_{\mathrm{N}}\,, \tag{5.8}$$

where $S^a{}_b$ is a constant real matrix. Here x^a and x^b_{N} are the old and new coordinates ($a, b = 2, \ldots, 5$), respectively. Such a transformation is equivalent to a similarity transformation on χ ($\chi = S\chi_{\mathrm{N}}S^{-1}$). Solutions related by such transformations actually belong to the same class of equivalence. However, when some Killing vectors have closed orbits, say ${\xi_2}^A$ and ${\xi_5}^A$ in our case, it is possible to generate new solutions which are not globally equivalent to old ones, as was shown in Section 4 of Ref. [AzCl96] and in Ref. [Az08].

Equations (5.6 and 5.7) have been derived under the sole ansatz (5.1), which is the general form of a stationary cylindrically symmetric 5-metric. Besides the property discussed in the previous paragraph, the system (5.6 and 5.7) possesses two further properties: if one performs the simultaneous transformations $\chi \to -\chi$ and $\rho \to -\rho$ the system (5.6 and 5.7) remains invariant, and the other property is that any solution χ to (5.6 and 5.7) commutes with its derivative $\chi_{,\rho}$: $[\chi, \chi_{,\rho}] = 0$.

In order to show that $[\chi, \chi_{,\rho}] = 0$, $\forall \rho$, we proceed as follows. Multiplying (5.6) from the left and from the right by χ, subtracting the two equations and using the fact that $[\chi, Q] = 0$, one obtains the equation $[\chi, Q_{,\rho}] = 0$ which is split as $(2 - \gamma G)[\chi, \chi_{,\rho}] - \gamma f[\chi, (\chi^2)_{,\rho}] + \gamma[\chi, (\chi^3)_{,\rho}] = 0$. Using the identity $[\chi, (\chi^n)_{,\rho}] \equiv [\chi^n, \chi_{,\rho}]$, ($n$ a positive integer), the latter equation reads

$$(2 - \gamma G)[\chi, \chi_{,\rho}] - \gamma f[\chi^2, \chi_{,\rho}] + \gamma[\chi^3, \chi_{,\rho}] = [Q, \chi_{,\rho}] = 0\,, \tag{5.9}$$

where we have used $[f, \chi_{,\rho}] \equiv 0$ in the last commutator of (5.9). By virtue of (5.9), the matrix $\chi_{,\rho}$ commutes then with any power of Q and consequently with any polynomial in Q. Hence, to complete the proof of $[\chi, \chi_{,\rho}] = 0$, we have to show that χ can be expressed as a polynomial in Q by inverting the *definition* formula (5.4). Squaring and cubing both sides of (5.4) and using (5.5) to eliminate any power of χ higher than 3, one obtains

$$Q^2 \stackrel{\text{def}}{=} P_0 + P_1\chi + P_2\chi^2 + P_3\chi^3\,, \tag{5.10}$$

$$Q^3 \stackrel{\text{def}}{=} \hat{P}_0 + \hat{P}_1\chi + \hat{P}_2\chi^2 + \hat{P}_3\chi^3\,, \tag{5.11}$$

where $P_0(\rho) \to P_3(\rho)$ and $\hat{P}_0(\rho) \to \hat{P}_3(\rho)$ are scalar polynomials of (f, G, H, k) which we obtained using MATLAB; only two of which are shown below:

$$P_0 = 16f^2 + (3/2)\gamma^2 Gk - 4\gamma k\,;\; P_3 = (1/3)\gamma^2 H + 8\gamma f - (1/2)\gamma^2 Gf\,.$$

In the generic case χ, χ^2, and χ^3 are seen as independent variables. Hence, the *linear* system of equations (5.4, 5.10, and 5.11) in the variables (χ, χ^2, χ^3) can be solved for any one of them. Let $G_{m,n} = m\gamma G - n$ (m, n are positive integers). If $L(\rho)$ is the determinant of the system of equations (5.4, 5.10, and 5.11),

$$\begin{aligned} L = &-288\gamma^6 k^3 - 288\gamma^4 G_{1,4}k^2 \\ &+ 6G_{1,4}^2\{\gamma[12G_{1,2}f^2 - 4\gamma fH + 3GG_{3,16}] + 48\}\gamma^2 k \\ &+ G_{1,4}^3\{4\gamma^3 H^2 - 6fG_{3,4}\gamma^2 H - 9G_{1,2}[\gamma G(G_{1,8} - 2\gamma f^2) + 16]\}\,, \end{aligned}$$

then χ is provided by

$$\begin{aligned}
\chi = \big\{&72\gamma(G_{1,4}^2 - 4\gamma^2 k)Q^3 + \gamma[72G_{1,4}^2(2fG_{1,7} - \gamma H) \\
&+ 96\,\gamma^2(2\gamma H - 3fG_{1,12})k]Q^2 + \big\{G_{1,4}^2[9G_{1,4}^3 + 18\gamma f^2(256 + \gamma GG_{3,68}) \\
&- 4\gamma^2[18\gamma f^2 G_{1,24}G_{1,8} + 9G_{1,4}^2 G_{5,16} - 12\gamma^2 fG_{5,52}H + 16\gamma^3 H^2] \\
&- 24\gamma^2 fG_{1,16}H + 8\gamma^3 H^2]k + 576\gamma^4 G_{1,3}k^2\big\}\,Q \\
&+ 4\{fG_{1,4}^2\,[12f\gamma^2 G_{5,28}H - 16\gamma^3 H^2 - 9G_{1,4}[16 + \gamma(6f^2 + G)G_{1,8}]] \\
&+ \gamma^2 k[72f^3\gamma G_{1,8}^2 - 96\gamma^2 f^2 G_{1,8}H + 4f(8\gamma^3 H^2 - 9G_{1,5}G_{1,4}^3) \\
&+ 3\gamma G_{1,4}^2 G_{5,12}H] + 6\gamma^4[3f(112 + \gamma GG_{3,44}) - 4\gamma G_{1,2}H]k^2 - 72\gamma^6 fk^3\}\}\,/L\,,
\end{aligned} \tag{5.12}$$

and similar results for χ^2 and χ^3

$$\chi^2 = [\gamma^2(48\gamma GH - 72\gamma GG_{1,4} - 192H - 288\gamma fk)Q^3 + \cdots]/L\,, \tag{5.13}$$

$$\begin{aligned}
\chi^3/24 = [&96G_{1,1} + 3\gamma^2 G^2 G_{1,10} - 12\gamma^2 kG_{1,2} - 12\gamma^3 f^2 k \\
&+ \gamma^2 fG_{1,4}(2H - 3fG)]Q^3 + \cdots]/L\,.
\end{aligned} \tag{5.14}$$

Using MATLAB, we have checked that the square and cube of the right-hand side of (5.12) coincide with the right-hand sides of (5.13) and of (5.14), respectively.

Now, for the values of ρ such that $L(\rho) \neq 0$, $\chi(\rho)$ is a polynomial in $Q(\rho)$ provided by (5.12), and since by (5.9) $\chi_{,\rho}(\rho)$ commutes with $Q(\rho)$, we conclude that $\chi_{,\rho}(\rho)$ commutes with $\chi(\rho)$. Since we are only interested in smooth solutions $\chi(\rho)$, the commutator $[\chi, \chi_{,\rho}](\rho)$ is also seen as a smooth continuous matrix function of ρ, so by continuity we extend the property $[\chi, \chi_{,\rho}] = 0$ to all values of ρ including the roots of $L(\rho) = 0$, if there are any. These statements being made for a fixed value of γ are extended by continuity to all values of γ. Hence, the commutator $[\chi, \chi_{,\rho}]$ vanishes *identically* for any solution $\chi(\rho, \gamma)$ to the system (5.6 and 5.7):

$$[\chi, \chi_{,\rho}](\rho, \gamma) \equiv 0\,.$$

We can now expand the left-hand side of (5.6) in such a way that the terms including $\chi_{,\rho}$ are grouped and $\chi_{,\rho}$ is factored, say, to the right of the powers of χ. Assume that $\chi(\rho)$ is *any* given solution to (5.6) and let $a(\rho)$, $b(\rho)$, $e(\rho)$, $d(\rho)$, $m(\rho)$, $n(\rho)$, $s(\rho)$, and $t(\rho)$ be eight scalar real functions. We want to determine under which condition(s) the product of the matrix $m+n\chi+s\chi^2+t\chi^3$ by the left-hand side of (5.6) is *identically* a total derivative. Mathematically speaking, given *any* solution χ to (5.6) we want to determine the equations satisfied by the eight functions $a(\rho) \to t(\rho)$ and the conditions of their resolutions such that

$$(m + n\chi + s\chi^2 + t\chi^3) \times [\text{l.h.s of (5.6)}] \equiv (a + b\chi + e\chi^2 + d\chi^3)_{,\rho}\,. \tag{5.15}$$

Now, both sides of (5.15) being *identically* equal for *any* solution χ to (5.6) leads to eight equations; four are algebraic and the other four are differential equations. The algebraic equations are the coefficients of $\chi^3\chi_{,\rho}$, $\chi^2\chi_{,\rho}$, $\chi\chi_{,\rho}$, and $\chi_{,\rho}$ on both sides of (5.15) expressing e, d, n, and b in terms of $(t,\, s,\, m)$. Because of limited space, we only show the expressions of (n, b)

$$6\gamma n = -(4+\gamma G+2\gamma f^2)t - 2\gamma f s\,;\; b = -2\gamma f k t - 6\gamma k s + (4-2\gamma G)m\,, \quad (5.16)$$

where k is provided by (5.7). Using these algebraic relations in the other four differential equations, which are the coefficients of χ^3, χ^2, χ, and the independent terms on both sides of (5.15), we obtain the *linear* differential equations satisfied by $(t,\, s,\, m,\, a)$, where only one of them is shown below:

$$\begin{aligned} &- a_{,\rho} + [\gamma f^3 k/3 + 5\gamma G f k/3 + 2\gamma f_{,\rho} f k - 4 f k/3 + \gamma G_{,\rho} k]t \\ &+ [\gamma f^2 k/3 + 2\gamma f_{,\rho} k + \gamma G k]s + [8 f_{,\rho} + 2G + 4 f^2]m = 0\,. \end{aligned}$$

Since these four differential equations are linear in $(t,\, s,\, m,\, a)$, we are guaranteed that solutions always exist. If any solution $\chi(\rho)$ is known, its invariants $(f,\, G,\, H,\, k)$ can be substituted in these differential equations and solutions, at least in the form of power series or hypergeometric functions for the unknowns $(t,\, s,\, m,\, a)$, can be derived. Using the algebraic equations [(5.16), ...], one determines the remaining four unknowns $(e,\, d,\, n,\, b)$.

Now, given *any* solution χ to (5.6 and 5.7), assume that the eight functions $a(\rho) \to t(\rho)$ have been determined as described previously. Multiplying both sides of (5.6) by the matrix $m + n\chi + s\chi^2 + t\chi^3$ and using (5.15), one obtains

$$(a + b\chi + e\chi^2 + d\chi^3)_{,\rho} = 0 \quad \Rightarrow \quad a + b\chi + e\chi^2 + d\chi^3 = A\,, \qquad (5.17)$$

where A is a 4×4 constant real matrix. One then should be able to invert the second equation in (5.17) and express χ as a polynomial in A by applying a similar procedure as in the steps from (5.10) to (5.12) to obtain

$$\chi(\rho) = \eta(\rho) + \omega(\rho)A + \beta(\rho)A^2 + \delta(\rho)A^3\,. \qquad (5.18)$$

Hence, *any* solution χ to (5.6) is necessarily a polynomial in a constant real matrix A with scalar coefficient functions of ρ.

Solutions to the system (5.6 and 5.7) will be grouped according to their determinant. In Section 5.4 we will somehow rely on our previous exact solutions for the case $\gamma = 0$ (corresponding to pure KKT without GBT) [AzCl96], which will not be discussed here. Our results for $\gamma = 0$ are summarized in (5.19) and (5.20) where A is a 4×4 constant real matrix with arbitrary $\mathrm{tr}\, A^3$ and $\det A$:

$$\chi = A\,, \quad \text{with } \mathrm{tr}\, A = \mathrm{tr}\, A^2 = 0 \; (\gamma = 0)\,; \qquad (5.19)$$

$$\chi = (2/\rho)A\,, \quad \text{with } \mathrm{tr}\, A = \mathrm{tr}\, A^2 = 1 \; (\gamma = 0)\,. \qquad (5.20)$$

5.3 Singular Solutions: $k \equiv 0$

Solutions with vanishing determinant (5.7) satisfy either

$$G = 0 \quad \text{or} \quad 8 + 2\gamma G + 3\gamma f^2 + 2\gamma f_{,\rho} = 0\,. \tag{5.21}$$

For $G = 0$ equation (5.6) reduces to $2Q_{,\rho} + fQ = 0$, whose solution is given by

$$Q(\rho) = \exp[-F(\rho)]M\,, \quad \text{with} \quad F(\rho) = (1/2)\int^{\rho} f(\rho')\ \mathrm{d}\rho' \tag{5.22}$$

and M is a 4×4 constant real matrix. Using (5.4) in (5.22) one obtains

$$M - 4f\exp[F] = (2\chi - \gamma f\chi^2 + \gamma\chi^3)\exp[F]\,. \tag{5.23}$$

With $k = 0$, the determinant of the right-hand side of (5.23) is zero, and consequently $4f\exp[F]$ must be a constant (identified with one of the eigenvalues of M). Hence, $\{4f\exp[F]\}_{,\rho} = 0$ leads to $f = 0$ or $f = 2/(\rho - \rho_0)$. The trivial case $f = 0$ leads to the following solution where A is a constant matrix:

$$\chi = A\,, \ \text{ with } \ \mathrm{tr}\,A = \mathrm{tr}\,A^2 = \det A = 0\,.$$

For the case $f = 2/\rho$ (we take $\rho_0 = 0$), equation (5.23) reduces to

$$4A = 2\rho\chi - 2\gamma\chi^2 + \gamma\rho\chi^3\,, \tag{5.24}$$

where $A = (M - 8)/4$. Inverting equation (5.24) by applying the same steps from (5.10) to (5.12), one obtains χ as a function of A:

$$\chi(\rho) = (2/\rho)A - (4\gamma/\rho^3)(A^3 - A^2)\,, \qquad \text{with } \ \mathrm{tr}\,A = \mathrm{tr}\,A^2 = \mathrm{tr}\,A^3 = 1 \text{ and } \det A = 0\,. \tag{5.25}$$

In the following we will discuss the classification of the solutions (5.25). One derives the 5-metric (5.1) upon integrating $\chi = \lambda^{-1}\lambda_{,\rho}$:

$$\lambda = C\{1 - A^3 + \rho^2 A^3 - 2\ln\rho(A^3 - A) - 2[(\ln\rho)^2 - \gamma/\rho^2](A^3 - A^2)\}\,, \tag{5.26}$$

where C is a constant real matrix of signature $(--+-)$. λ being symmetric, C satisfies the relations $C = C^{\mathrm{T}}$ and $CA = (CA)^{\mathrm{T}}$ ($^{\mathrm{T}}$ denotes transpose). Equation (5.26) leads to $\det\lambda = (\det C)\rho^2$; hence, the 5-metric is singular along the axis $\rho = 0$, and consequently ρ runs from 0 to ∞.

The solutions (5.25) and (5.26) are in their generic forms. The constraints on A, which fix the invariants of A, do not fix its rank $\mathrm{r}(A)$. Hence, solutions provided by (5.26) can be classified according to their rank. Notice that the rank of a matrix is invariant under a similarity transformation [LaTi85], such as that defined in (5.8). Consequently, the different solutions classified according to their rank belong to different equivalence classes. The constraints on A reduce its characteristic equation to $A^4 = A^3$. A has then the eigenvalues 1, 0, 0, and 0 without necessarily being diagonalizable. The simplest form to which one can bring the matrix A is the Jordan normal form [LaTi85],

$$A = (1,\ 0,\ 0,\ 0;\ \ 0,\ 0,\ \epsilon_3,\ 0;\ \ 0,\ 0,\ 0,\ \epsilon_4;\ \ 0,\ 0,\ 0,\ 0), \tag{5.27}$$

where $\epsilon_3,\ \epsilon_4 = 0$ or 1. The rank of A depends on the number of Jordan blocks associated with the eigenvalue 0 in (5.27). The special solutions $\chi = (2/\rho)A$, of rank 1 ($A^2 = A$) or 2 ($A^3 = A^2$), are interpreted as neutral or charged cosmic strings, respectively. The generic solution (5.25) of rank 3 ($A^3 \neq A^2$) is interpreted as a superconducting cosmic string [AzCl96, Az08].

Although the derived solutions (5.25) are in their generic form, they can be generalized by performing a similarity transformation (5.8) which results in new solutions. Since two of our Killing vectors, ${\xi_2}^A$ and ${\xi_5}^A$, have closed orbits, it is possible to generate new solutions which are not globally equivalent to old ones if at least one of these two vectors is rescaled or mixed with the other vectors as a result of the transformation (5.8): i.e., if ${S^5}_5 \neq 1$ and/or ${S^4}_5 \neq 0$. From this perspective, three examples have been given, two in Ref. [AzCl96] and one in Ref. [Az08], where we have generalized the superconducting cosmic string.

A thorough treatment of the case (5.21): $8 + 2\gamma G + 3\gamma f^2 + 2\gamma f_{,\rho} = 0$ is possible, leading to no solution to equations (5.6 and 5.7). Alternatively, one refers to the section on regular solutions, which includes this case as a special one.

5.4 Regular Solutions: $k \neq 0$

For convenience we reparametrize the diagonal elements of χ [equation (5.18)] by $T_a(\rho) \equiv \eta(\rho) + p_a\omega(\rho) + p_a^2\beta(\rho) + p_a^3\delta(\rho)$, where the p_a's are the eigenvalues of A ($a = 2 \cdots 5$). If two or more p_a's are equal, the corresponding T_a's are equal too; in any case, the number of independent equations satisfied by T_a's [equations (5.6 and 5.7)] exceeds by one that of independent functions $T_a(\rho)$.

The equations satisfied by T_a's are the diagonal elements of (5.6) and (5.7),

$$\begin{aligned}
&[4 + 6\gamma T_a^2 - 2\gamma G - 4\gamma f T_a]T_{a,\rho} + 8f_{,\rho} - 2\gamma T_a^2 f_{,\rho} - \gamma T_a G_{,\rho} \\
&+ 4f^2 + 2fT_a + 2G - \gamma G T_a^2 - \gamma f G T_a - \gamma f^2 T_a^2 + \gamma f T_a^3 = 0\,; \quad (5.28)\\
&24\gamma k = G(8 + 2\gamma G + 3\gamma f^2 + 2\gamma f_{,\rho})\,, \quad (5.29)
\end{aligned}$$

where $k(\rho) = \prod_{a=2}^{5} T_a(\rho)$, $f(\rho) = \sum_{a=2}^{5} T_a(\rho)$, and $G(\rho) = -\sum_{\substack{a,\,b=2\\ b\neq a}}^{5} T_a(\rho)T_b(\rho)$. In the generic case where all T_a's are non-equal, a solution to (5.28) represents a (hyper)curve (C) in the four-dimensional space of coordinates T_a, where ρ is an affine parameter. Equations (5.28) are linear in $T_{a,\rho}$'s and can be solved for the latter in terms of T_a's then used in (5.29) to eliminate $f_{,\rho}$. The remaining algebraic equation (5.29) in T_a represents a hypersurface (S) in the above-mentioned four-dimensional space. Hence, the system (5.28 and 5.29) will admit a solution only if the curve (C) or a segment of it lies on the hypersurface (S). The purpose of the following is to show that this is not the case.

We will make use of (5.18), where A is any constant matrix, and expand the functions $(\eta(\rho),\dots,\delta(\rho))$ by power series in $1/\rho$: $\eta=\sum_{i=0}\eta_i/\rho^i$, $\dots$, $\delta=\sum_{i=0}\delta_i/\rho^i$, where $(\eta_i,\dots,\delta_i)$ are numerical constants. Substituting into equation (5.18), one writes

$$\chi=\sum_{i=0}M_i/\rho^i=M_0+M_1/\rho+M_2/\rho^2+\cdots\,, \tag{5.30}$$

where $M_i=\eta_i+\omega_iA+\beta_iA^2+\delta_iA^3$ $(i\geq 0)$ are all commuting constant matrices since they are polynomials of the same constant matrix A. Hence, we will look for solutions of the form (5.30) where M_i [$i\geq 1$ case (5.19) and $i\geq 2$ case (5.20)] are *smooth* functions of γ which vanish in the limit $\gamma\to 0$. Two cases are to be distinguished: $M_0\neq 0$ and $M_0=0$ corresponding to (5.19) and (5.20), respectively.

Notice that solutions of the form $\chi=\rho^n(N_0+N_1/\rho+N_2/\rho^2+\cdots)$ where N_i are constant matrices and $n\in\mathbb{N}^+$, which diverge at spatial infinity ($\rho\to\infty$), do not exist. This is because when the field equations (5.6 and 5.7) are satisfied, the vanishing of the coefficients of the leading terms in the series expansions of (5.6 and 5.7) leads to the vanishing of the leading term in the above expansion, i.e., $N_0=0$, and so on until all N_i are zero for $i\leq n-1$.

Since the matrices M_i commute, we introduce a simplified notation for the traces of their products, which will serve later to implement Mathematica-based symbolic evaluations. The blank between the symbols "tr" and "M" is removed, and the face of the symbol "M" is upright. For instance, $\mathrm{tr}\,(M_1M_2^2)$ is written as $\mathrm{trM}_{1,2,2}$ and $\mathrm{tr}\,(M_1M_5^3M_4)$ as $\mathrm{trM}_{1,4,5,5,5}$.

5.4.1 The Case $M_0\neq 0$

In the limit $\gamma\to 0$ the matrix χ, as given by (5.30), approaches a constant matrix M_0 which solves the equations of the pure KKT. Hence, M_0 satisfies the constraints [see (5.19)]: $\mathrm{trM}_0=\mathrm{trM}_{0,0}=0$ with arbitrary $\mathrm{trM}_{0,0,0}\equiv P$ and detM_0. To determine the remaining matrices M_i, $i\geq 1$, we proceed by induction. Let us assume that all the matrices M_i for $1\leq i\leq l-1$ are zero and look for the matrix of order l:

$$\chi=M_0+M_l/\rho^l+M_{l+1}/\rho^{l+1}+\cdots\,. \tag{5.31}$$

Substituting (5.31) into (5.7), the independent term leads immediately to $\mathrm{detM}_0=0$, and the matrix coefficient of $1/\rho^l$ in (5.6) is written as

$$\mathrm{trM}_l(2M_0+\gamma M_0^3)-2\gamma\mathrm{trM}_{0,l}M_0^2=-4\mathrm{trM}_{0,l}\,. \tag{5.32}$$

The determinant of (5.32) leads to $\mathrm{trM}_{0,l}=0$ and since $M_0\neq 0$ implies $2M_0+\gamma M_0^3\neq 0$ ($\Rightarrow P\neq 0$), the remaining equation (5.32) leads to $\mathrm{trM}_l=0$. Now, with $\mathrm{trM}_{0,l}=0$ the scalar coefficient of $1/\rho^l$ in (5.7) vanishes identically, and the series expansion of (5.5) to the order l implies

$$M_l = (\mathrm{trM}_{0,0,l}/P)M_0\,. \tag{5.33}$$

Next, evaluating the determinant of the matrix coefficient of $1/\rho^{l+1}$ in (5.6) implies $\mathrm{trM}_{0,l+1} = 0$, and the remaining coefficient reduces to

$$2\underbrace{\left(l\,\frac{\mathrm{trM}_{0,0,l}}{P} - \mathrm{trM}_{l+1}\right)}_{\Pi_1} M_0 + \gamma\underbrace{\left(6l\,\frac{\mathrm{trM}_{0,0,l}}{P} - \mathrm{trM}_{l+1}\right)}_{\Pi_2} M_0^3 = 0\,. \tag{5.34}$$

Tracing this last equation, we obtain $\Pi_2 = 0$ ($P \neq 0$) and the equation reduces to $2\Pi_1 M_0 = 0$; with $M_0 \neq 0$ this implies $\Pi_1 = 0$. The homogeneous system of equations $\Pi_1 = 0$ and $\Pi_2 = 0$ admits the trivial and unique solution $\mathrm{trM}_{l+1} = 0$ and $\mathrm{trM}_{0,0,l} = 0$. Hence, $M_l = 0$ by (5.33). Notice that all the equations and steps from (5.32) to (5.34) are valid for $l \geq 1$. Repeating these steps for $l = 1$ leads to $M_1 = 0$, then for $l = 2$ leads to $M_2 = 0$ and so on.

We have thus shown that $\chi = M_0 \neq 0$ with $\mathrm{trM}_0 = \mathrm{trM}_{0,0} = \mathrm{detM}_0 = 0$ and $\mathrm{trM}_{0,0,0}$ arbitrary is the unique solution of the form (5.30). Said otherwise, solutions of the form (5.30) with $\det\chi \neq 0$ and $M_0 \neq 0$ do not exist.

5.4.2 The Case $M_0 = 0$

In the limit $\gamma \to 0$ the matrix χ, as given by (5.30), approaches the matrix M_1/ρ which solves the equations of the pure KKT. Hence, M_1 satisfies the constraints [see (5.20)]: $\mathrm{trM}_1 = 2$ and $\mathrm{trM}_{1,1} = 4$ with arbitrary $\mathrm{trM}_{1,1,1}$ and detM_1. The coefficients of $1/\rho^3$ in (5.7) and (5.6) lead, respectively, to $\mathrm{trM}_{1,2} = 2\mathrm{trM}_2$ and $M_2 = (\mathrm{trM}_2/2)M_1$. In general, the coefficients of $1/\rho^{l+1}$ in (5.7) and (5.6) lead to scalar and matrix equations, respectively, depending linearly on trM_l and $\mathrm{trM}_{1,l}$. Furthermore, the matrix equation depends linearly on M_l. Such a system can always be solved for (trM_l, $\mathrm{trM}_{1,l}$, M_l). Since the resolution of the system involves tracing the matrix equation, consistency of the obtained solution (trM_l, $\mathrm{trM}_{1,l}$, M_l) has to be checked for each step. For instance, the coefficients of $1/\rho^4$ in (5.7) and (5.6) lead to

$$3\gamma\mathrm{detM}_1 + 4\mathrm{trM}_3 - 2\mathrm{trM}_{1,3} = 0 \tag{5.35}$$

$$\begin{aligned} &4\gamma(2M_1^2 - M_1^3) + (\mathrm{trM}_2^2 + 2\mathrm{trM}_3)M_1 - 8M_3 \\ &+ 4\mathrm{trM}_2^2 - 16\mathrm{trM}_3 + 4\mathrm{trM}_{1,3} = 0\,. \end{aligned} \tag{5.36}$$

Solving the system consisting of (5.35) and the trace of (5.36), we obtain

$$\mathrm{trM}_3 = \frac{2}{3}\gamma\mathrm{detM}_1 + \frac{1}{2}\mathrm{trM}_2^2 + \frac{\gamma}{9}(8 - \mathrm{trM}_{1,1,1}) \tag{5.37}$$

$$\mathrm{trM}_{1,3} = \frac{17}{6}\gamma\mathrm{detM}_1 + \mathrm{trM}_2^2 + \frac{2\gamma}{9}(8 - \mathrm{trM}_{1,1,1})\,. \tag{5.38}$$

Substituting these last two equations into (5.36), we obtain

$$M_3 = \frac{\gamma}{2}(2M_1^2 - M_1^3) + \frac{1}{4}[\mathrm{trM}_2^2 + \frac{2\gamma}{3}\mathrm{detM}_1 + \frac{\gamma}{9}(8 - \mathrm{trM}_{1,1,1})]M_1 + \frac{\gamma}{12}\mathrm{detM}_1 - \frac{\gamma}{9}(8 - \mathrm{trM}_{1,1,1})\,. \quad (5.39)$$

Tracing (5.39) reduces to (5.37) and tracing (5.39)$\times M_1$ reduces to (5.38). So the solution (trM_3, $\mathrm{trM}_{1,3}$, M_3) is consistent. Similarly, we obtained a consistent solution (trM_4, $\mathrm{trM}_{1,4}$, M_4); however, the solution (trM_5, $\mathrm{trM}_{1,5}$, M_5) failed to be consistent. The inconsistency of (trM_5, $\mathrm{trM}_{1,5}$, M_5) leads to two constraints on the free parameters ($\mathrm{trM}_{1,1,1}$, detM_1, trM_2):

$$45\gamma\mathrm{detM}_1^2 + [216\mathrm{trM}_2^2 - 13\gamma(-8 + \mathrm{trM}_{1,1,1})][(-8 + \mathrm{trM}_{1,1,1}) + 6\mathrm{detM}_1[27\mathrm{trM}_2^2 + \gamma(-37 + 8\mathrm{trM}_{1,1,1})] = 0 \quad (5.40)$$

$$90\gamma\mathrm{detM}_1^2 + 8[54\mathrm{trM}_2^2 - \gamma(-8 + \mathrm{trM}_{1,1,1})](-8 + \mathrm{trM}_{1,1,1}) + 3\mathrm{detM}_1[108\mathrm{trM}_2^2 + \gamma(-496 + 89\mathrm{trM}_{1,1,1})] = 0\,. \quad (5.41)$$

Other constraints on ($\mathrm{trM}_{1,1,1}$, detM_1, trM_2) are derived from the inconsistency of (trM_6, $\mathrm{trM}_{1,6}$, M_6), from that of (trM_7, $\mathrm{trM}_{1,7}$, M_7) or, preferably, from the traces of the matrix coefficients of $1/\rho^6$ and $1/\rho^7$ in (5.5):

$$[18\gamma\mathrm{detM}_1 - 9\mathrm{trM}_2^2 + 5\gamma(-8 + \mathrm{trM}_{1,1,1})](-8 + \mathrm{trM}_{1,1,1}) = 0 \quad (5.42)$$

$$\mathrm{trM}_2[18\gamma\mathrm{detM}_1 - 6\mathrm{trM}_2^2 + 5\gamma(-8 + \mathrm{trM}_{1,1,1})](-8 + \mathrm{trM}_{1,1,1}) = 0. \quad (5.43)$$

Applying the command `Reduce` of Mathematica to solve the system of the four constraints (5.40), (5.41), (5.42), and (5.43), with the extra conditions $\mathrm{detM}_1 \neq 0$ and $(\mathrm{detM}_1, \mathrm{trM}_2, \mathrm{trM}_{1,1,1}) \in \mathbb{R}$ to ensure that $\det\chi \neq 0$ and that the invariants of χ are real numbers, leads to the `False` result. This proves that solutions of the form (5.30) with $\det\chi \neq 0$ and $M_0 = 0$ do not exist.

From the above discussions, we then conclude that the system (5.6 and 5.7) does not admit any regular solution.

With different tools on hand, we have shown that the field equations (5.6) and (5.7) admit either 1) singular solutions of the form $\chi(\rho) = (2/\rho)A - (4\gamma/\rho^3)(A^3 - A^2)$ constrained by $\mathrm{tr}\,A = \mathrm{tr}\,A^2 = \mathrm{tr}\,A^3 = 1$ and $\det A = 0$. The integral constant matrix A helps to classify the solutions according to its rank. The outcome of this classification is that solutions with $\mathrm{r}(A) = 1$, $\mathrm{r}(A) = 2$, and $\mathrm{r}(A) = 3$ are neutral, charged, and superconducting cosmic strings, respectively, or 2) singular solutions of the form $\chi = A$ constrained by $\mathrm{tr}\,A = \mathrm{tr}\,A^2 = \det A = 0$. In Section 5.4 we have conducted proofs of nonexistence of regular and further singular solutions to the overdetermined system of nonlinear differential equations (5.6) and (5.7).

A program for Mathematica has been developed to deal with commuting matrices in algebraic form instead of the usual matrix form. It consists in evaluating traces of products of matrices and determinants of sums of matrices in algebraic forms without however knowing the entries of the matrices.

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6

The Bottom of the Spectrum in a Double-Contrast Periodic Model

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A periodic spectral problem in a bounded domain with double inhomogeneities in mass density and stiffness coefficients is considered. A previous study [BKS08] has explored the problem by the method of asymptotic expansions with justification of errors showing that all eigenelements of the homogenized problem really approximate some of the perturbed eigenelements. Within this chapter additional results, partly announced in [BKS08], are obtained on the eigenfunction convergence at the bottom of the spectrum. It is shown that the eigenfunctions, which correspond to the eigenvalues at the bottom of the spectrum, could converge either to zero or to the eigenfunctions of the homogenized problem. The result was obtained by the method of two-scale convergence [Al92, Zh00].

Similar double high contrasts in mass and stiffness coefficients for a finite number of perturbed regions were considered in [BaGo07]–[BaGo09] and [GLNP06]. One of the distinctive features of these models is the presence of two different types of eigenvibrations at low and high frequencies when particular subdomains generate leading frequencies and eigenvibrations. Comparing the results for the same relative magnitude of perturbations, it is observed that in the case of only two perturbed regions, low-frequency vibrations are generated by the heavier inclusion. This is not the case in the periodic model under consideration, where even at low frequencies the homogenized problem in a relatively light matrix appears. Nevertheless, the presence of small periodic heavy inclusions of order ε^{-1} shifts the bottom of the spectrum itself, inducing an eigenvalue series of order ε, in particular $c\varepsilon \leq \lambda_1^\varepsilon \leq \varepsilon C$ (see Lemma 5). Considering highly nontrivial effects appearing in periodic problems with high contrasts [Al92, JKO94], we refer to [BeGr05, Pa91, Ry02, Sa98, Sa99] for the specific features arising within models including mass density perturbations.

We consider a model of eigenvibrations for a body occupying a bounded domain Ω in $\mathbb{R}^n$ $(n = 2, 3, \dots)$ containing a periodic array of small inclusions;

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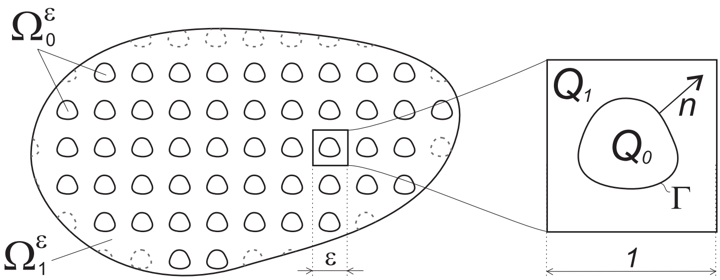

Fig. 6.1. The geometry and the periodicity cell

see Figure 6.1. The size of the inclusions is controlled by a small positive parameter ε, $\varepsilon \to 0$.

Let $Q = [0,1)^n$ be a reference periodicity cell in $\mathbb{R}^n$. Let $\widetilde{Q}_0$ be a periodic set of "inclusions," i.e., $\widetilde{Q}_0 + m = \widetilde{Q}_0$, $\forall m \in \mathbb{Z}^n$, and $Q_0 = \widetilde{Q}_0 \cap Q$ is a reference inclusion lying inside Q ($\overline{Q_0} \subset Q$ with the bar denoting a closure of the set) with C^2-smooth boundary Γ; see Figure 6.1. Let $Q_1 = Q \backslash \overline{Q_0}$, $\widetilde{Q}_1 = \mathbb{R}^n \backslash \overline{\widetilde{Q}_0}$. Introducing $y = x/\varepsilon$ we refer to y as a fast variable, as opposed to the slow variable x. In the x-variable the periodicity cell is $\varepsilon Q = [0,\varepsilon)^n$. If $y \in Q_j$ then $x = \varepsilon y \in \varepsilon Q_j$, $j = 0,1$. We denote $\tilde{\Omega}_0^\varepsilon := \Omega \cap \varepsilon\tilde{Q}_0$, $\tilde{\Omega}_1^\varepsilon := \Omega \cap \varepsilon\tilde{Q}_1$; see Figure 6.1. Within this chapter two possible geometries are under consideration:

A The inclusions are allowed to intersect or touch the boundary; then simply $\Omega_k^\varepsilon := \tilde{\Omega}_k^\varepsilon$, $k = 0,1$.

B The inclusions touching or intersecting the boundary are sent to the connected phase: if the intersection between $\partial\Omega$ and the boundary of any connected component of $\tilde{\Omega}_0^\varepsilon$ is nonempty, then this particular component is sent to be part of a new matrix $\Omega_1^\varepsilon \supset \tilde{\Omega}_1^\varepsilon$, and the remaining components form a new $\Omega_0^\varepsilon \subset \tilde{\Omega}_0^\varepsilon$.

Let Γ^ε be a boundary between Ω_0^ε and Ω_1^ε. The trace on Γ^ε of function $f : \Omega_j^\varepsilon \to \mathbb{R}^n$ is denoted by $f\big|_j$. Let n_y be the outer unit normal to Q_0 on its boundary Γ, and let n_x denote the similar normal on Γ^ε.

Let the stiffness a_ε and density ρ_ε be parametrized by $\varepsilon > 0$ as follows:

$$a_\varepsilon(x) = \begin{cases} 1, & x \in \Omega_1^\varepsilon \\ \varepsilon, & x \in \Omega_0^\varepsilon \end{cases} \quad \text{and} \quad \rho_\varepsilon(x) = \begin{cases} 1, & x \in \Omega_1^\varepsilon \\ \varepsilon^{-1}, & x \in \Omega_0^\varepsilon \end{cases}.$$

We study the asymptotic behavior of the self-adjoint spectral problem

$$\int_\Omega a_\varepsilon(x) \nabla u_\varepsilon \nabla\phi \, dx - \lambda^\varepsilon \int_\Omega \rho_\varepsilon(x) u_\varepsilon \phi \, dx = 0 \quad \forall\phi \in H_0^1(\Omega) \tag{6.1}$$

as $\varepsilon \to 0$. If Γ and $\partial\Omega$ are smooth enough, then the variational problem (6.1) can be equivalently represented in a classical formulation,

$$-\text{div}\,(a_\varepsilon\,(x)\,\nabla u_\varepsilon) = \lambda^\varepsilon \rho_\varepsilon\,(x)\,u_\varepsilon, \quad x \in \Omega, \quad u_\varepsilon|_{\partial\Omega} = 0, \tag{6.2}$$

$$u_\varepsilon\big|_1 = u_\varepsilon\big|_0, \quad \partial_n u_\varepsilon\big|_1 = \varepsilon\partial_n u_\varepsilon\big|_0, \tag{6.3}$$

with symbol ∂_n denoting a derivative in the normal direction n, $\partial_n = n \cdot \nabla$.

We use the standard notation for Lebesgue and Sobolev spaces: $L^2_p(\Omega)$ is a p-weighted L^2-space of square integrable functions in Ω. The notation $(\cdot,\cdot)_H$ is used for a scalar product in a Hilbert space H. Let $\mathcal{L}^\varepsilon = L^2_{\rho_\varepsilon}(\Omega)$ and $\mathcal{H}^\varepsilon$ be an $H^1_0(\Omega)$ Sobolev space with scalar product

$$(u,v)_{\mathcal{H}^\varepsilon} = \int_\Omega a_\varepsilon(x)\nabla u \cdot \nabla v\,dx + \int_\Omega \rho_\varepsilon(x)uv\,dx.$$

The spectrum of (6.2), (6.3) consists of a countable set of eigenvalues of finite multiplicity with the only accumulation point at infinity:

$$0 < \lambda_1^\varepsilon < \lambda_2^\varepsilon \le \cdots \le \lambda_j^\varepsilon \le \cdots \to +\infty.$$

The corresponding eigenfunctions u_j^ε form an orthogonal basis in $\mathcal{L}^\varepsilon$:

$$0 = (u_j^\varepsilon, u_k^\varepsilon)_{\mathcal{L}^\varepsilon} = \int_{\Omega_1^\varepsilon} u_j^\varepsilon u_k^\varepsilon dx + \varepsilon^{-1}\int_{\Omega_0^\varepsilon} u_j^\varepsilon u_k^\varepsilon dx \quad \text{if} \quad j \ne k.$$

Then (6.1) shows that the eigenfunctions u_j^ε are orthogonal in $\mathcal{H}^\varepsilon$ as well,

$$0 = (u_j^\varepsilon, u_k^\varepsilon)_{\mathcal{H}^\varepsilon} = \int_{\Omega_1^\varepsilon} \nabla u_j^\varepsilon \cdot \nabla u_k^\varepsilon dx + \varepsilon\int_{\Omega_0^\varepsilon} \nabla u_j^\varepsilon \cdot \nabla u_k^\varepsilon dx \quad \text{if} \quad j \ne k.$$

Note that we do not fix the norm of u_j^ε yet. The reason is that different energy norms are more appropriate for the analysis of the problem at various energy levels, i.e., at various frequency scales.

We denote by $L^2_\#(Q)$ the space of functions in $L^2(Q)$ extended by Q-periodicity to the whole $\mathbb{R}^n$. Let $C^\infty_\#(Q)$ be the space of infinitely differentiable functions in $\mathbb{R}^n$ that are Q-periodic. Then $H^1_\#(Q)$ is the closure of $C^\infty_\#(Q)$ in the norm of $H^1(Q)$. Let V_{pot} be the space of potential vectors, i.e., vectors from the closure of the set $\{\nabla\phi|\ \ \phi \in C^\infty_\#(Q)\}$ in $L^2_\#(Q)^n$. Let V_{sol} be the space of solenoidal vectors, i.e., vectors b from $L^2_\#(Q)^n$ such that $\text{div}\, b = 0$ in $L^2_\#(Q)$. We also use the conventional notation $\phi(x,y) \in L^2(\Omega \times Q, H(Q))$ if function ϕ is from $L^2(\Omega \times Q)$ and, additionally, when it is considered as a function of the y-variable, $\phi(x,\cdot)$ belongs to a certain space $H(Q)$.

A previous study of the problem has discovered the presence of low frequencies, which correspond to the eigenvalues of order ε (specific case $\lambda_0 = 0$ within [BKS08, Th 4.6]). Moreover, for such eigenvalues the limit forms of vibrations also exist in a classical meaning, i.e., the limit forms in both phases depend only on the slow variable x and do not depend on the fast variable $y = x/\varepsilon$. Note that the latter statement does not hold true at high frequencies; see [BKS08], where the vibrations in the inclusions depend on the fast variable. Therefore, the desired result in the scope of this chapter is, first, to estimate the bottom of the spectrum for problem (6.1), which is addressed in Lemma 5, yielding the estimate $c\varepsilon \leq \lambda_1^\varepsilon \leq C\varepsilon$, and, second, to investigate the convergence of eigenfunction sequences u_ε corresponding to the low eigenvalues λ^ε subject to the conditions

$$\lambda^\varepsilon = \mathcal{O}(\varepsilon) \text{ as } \varepsilon \to 0, \quad \|u_\varepsilon\|_{L^2(\Omega)} = 1. \tag{6.4}$$

The latter question is addressed throughout the chapter and the results are gathered in Lemma 6.

Let the symbol $\overset{H}{\rightharpoonup}$ denote convergence in the weak topology of the Hilbert space H and $\overset{2}{\rightharpoonup}$ state for the weak two-scale convergence. Let χ_Ω, χ_j, and χ_j^ε denote the characteristic functions of the sets Ω, Q_j, and Ω_j^ε, respectively, $j = 0, 1$. Note that $\chi_j^\varepsilon(x) = \chi_\Omega(x)\chi_j(\frac{x}{\varepsilon})$.

Lemma 1. *Under assumptions* (6.4) *the sequence u_ε is uniformly bounded in $\mathcal{H}^\varepsilon$, i.e., there exists a constant $C > 0$ independent of ε and such that*

$$\|\nabla u_\varepsilon\|_{L^2(\Omega_1^\varepsilon)} \leq C, \quad \varepsilon^{1/2}\|\nabla u_\varepsilon\|_{L^2(\Omega_0^\varepsilon)} \leq C.$$

There exists a function $u(x) \in L^2(\Omega)$ such that up to a subsequence

$$u_\varepsilon \overset{L^2(\Omega)}{\rightharpoonup} u(x) \quad \textit{and} \quad u_\varepsilon \overset{2}{\rightharpoonup} u(x).$$

Proof. Let $\omega^\varepsilon = \varepsilon^{-1}\lambda^\varepsilon$. Then by virtue of (6.4), the sequence ω^ε is bounded. Integral identity (6.1) with $\phi = u_\varepsilon$ yields

$$\int_{\Omega_1^\varepsilon} |\nabla u_\varepsilon|^2 dx + \varepsilon \int_{\Omega_0^\varepsilon} |\nabla u_\varepsilon|^2 dx = \omega^\varepsilon \left(\varepsilon \int_{\Omega_1^\varepsilon} u_\varepsilon^2 dx + \int_{\Omega_0^\varepsilon} u_\varepsilon^2 dx \right).$$

Since u_ε is bounded in $L^2(\Omega)$, there exists a function $u(x) \in L^2(\Omega)$ such that up to a subsequence $u_\varepsilon \overset{L^2(\Omega)}{\rightharpoonup} u(x)$. Additionally, from the properties of two-scale convergence (see [Zh00, Prop. 2.2]), we have the existence of a function $u_0(x, y)$ such that $u_\varepsilon \overset{2}{\rightharpoonup} u_0(x, y)$. Moreover, by the mean value property, $u(x) = \langle u_0(x, \cdot)\rangle := \int_Q u_0(x, y)dy$. We introduce a measure $d\mu_\varepsilon = \chi_1^\varepsilon dx$. Since the measure $d\mu_\varepsilon$ is ergodic and $\varepsilon\|\nabla u_\varepsilon\|_{L^2(\Omega_1^\varepsilon)} \to 0$, by virtue of [Zh00, Th. 4.1] we obtain that $u_0(x, y)$ is a function of the slow variable x only. Then, naturally, $u_0(x, y) \equiv u(x)$.

Lemma 2. *The function u, which is defined in Lemma 1, belongs to $H_0^1(\Omega)$. There exists a function $u_1(x, y) \in L^2(\Omega, H_\#^1(Q))$ such that up to a subsequence $\chi_1^\varepsilon \nabla u_\varepsilon \overset{2}{\rightharpoonup} \chi_1(y)(\nabla u(x) + \nabla_y u_1(x, y))$.*

Proof. The proof mainly follows [Zh00, Proof of Th. 4.2]. Lemma 1 ensures that $\chi_1^\varepsilon \nabla u_\varepsilon$ is a bounded sequence in $L^2(\Omega)$. Therefore, up to a subsequence, it possesses a weak two-scale limit, which we denote by $p(x,y)$, i.e., $\chi_1^\varepsilon \nabla u_\varepsilon \xrightharpoonup{2} p(x,y)$ with $p(x,y) \in L^2(\Omega, L^2_\#(Q))$. Moreover, since $\chi_1(y)$ belongs to $L^\infty_\#(Q)$ and, by Lemma 1, $u_\varepsilon \xrightharpoonup{2} u(x)$, by the properties of two-scale convergence we obtain

$$\chi_1^\varepsilon(x)u_\varepsilon(x) = \chi_\Omega(x)\chi_1(\frac{x}{\varepsilon})u_\varepsilon(x) \xrightharpoonup{2} \chi_\Omega(x)\chi_1(y)u(x). \tag{6.5}$$

Let $b(y) \in V_{\text{sol}}$ and $\phi(x) \in C^\infty(\overline{\Omega})$. Since $\phi \nabla u_\varepsilon = \nabla(u_\varepsilon \phi) - u_\varepsilon \nabla \phi$ and b is orthogonal to all potential vectors,

$$\int_\Omega \phi(x)\chi_1^\varepsilon(x)\nabla u_\varepsilon(x) \cdot b(\frac{x}{\varepsilon})\,dx = -\int_\Omega \chi_1^\varepsilon(x)u_\varepsilon(x)\nabla\phi(x) \cdot b(\frac{x}{\varepsilon})\,dx.$$

Passing to the limit in the last identity and incorporating (6.5), we have

$$\int_\Omega \phi(x) \int_Q p(x,y) \cdot b(y)\,dy\,dx = -\int_\Omega u(x)\nabla\phi(x) \cdot \int_Q \chi_1(y)b(y)\,dy\,dx. \tag{6.6}$$

With an arbitrary $\phi \in C^\infty(\Omega)$ and a constant vector

$$\langle b \rangle_1 = \int_{Q_1} b(y)\,dy,$$

the latter leads to the distributional equality

$$\int_Q p(\cdot,y) \cdot b(y)\,dy = \langle b \rangle_1 \cdot \nabla u \quad \text{in} \quad L^2(\Omega). \tag{6.7}$$

Note that the range of all possible values of $\langle b \rangle_1$ as $b \in V_{\text{sol}}$ covers the entire $\mathbb{R}^n$. Therefore, the function ∇u belongs to $L^2(\Omega)^n$ itself and, thus, $u \in H^1(\Omega)$.

Then (6.6) for $\phi \in C_0^\infty(\Omega)$ yields

$$\int_\Omega \int_Q [p(x,y) - \nabla u(x)] \cdot \phi(x)b(y)\,dy\,dx = 0.$$

Since the linear span of the vector functions $\phi(x)b(y)$ is dense in $L^2(\Omega, V_{\text{sol}})$ and the orthogonal decomposition $L^2(\Omega \times Q)^n = L^2(\Omega, V_{\text{pot}}) \oplus L^2(\Omega, V_{\text{sol}})$ holds true, we obtain $p(x,\cdot) - \nabla u(x) \in L^2(\Omega, V_{\text{pot}})$. Therefore, there exists a function $u_1(x,y) \in L^2(\Omega, H^1_\#(Q))$ such that $p(x,\cdot) - \nabla u(x) = \nabla_y u_1(x,\cdot)$. Note that, by the construction, $p(x,y) = \chi_1(y)p(x,y)$. Indeed, $(\chi_1^\varepsilon)^2 = \chi_1^\varepsilon$ and, therefore, $\chi_1^\varepsilon \nabla u_\varepsilon = (\chi_1^\varepsilon)^2 u_\varepsilon \xrightharpoonup{2} \chi_1(y)p(x,y)$.

Let us finally show that u satisfies zero boundary conditions. Substituting $p(x,y) = \nabla u(x) + \nabla_y u_1(x,y)$ into (6.6), we obtain

$$\int_\Omega \phi(x)\nabla u \cdot \int_Q b(y)\,dy\,dx = -\int_\Omega u(x)\nabla\phi(x) \cdot \int_{Q_1} b(y)\,dy\,dx. \tag{6.8}$$

Let the function b have support in Q_1. Then, by (6.8),

$$\int_\Omega \phi(x)\nabla u(x)\,dx = -\int_\Omega u(x)\nabla\phi(x)\,dx, \quad \forall\phi\in C^\infty(\overline{\Omega}).$$

Integrating by parts, we obtain $\int_{\partial\Omega} u\,\partial_\nu\phi\,d\gamma = 0$, where ν is the unit normal to $\partial\Omega$. Since ϕ is an arbitrary smooth function, the trace of u to $\partial\Omega$ is zero. Thus, $u\in H^1_0(\Omega)$.

Lemma 3. *The function $u_1(x,y)$, which is introduced in Lemma 2, is a solution in $L^2(\Omega\times Q, H^1_\#(Q))$ to the problem*

$$-\Delta_y u_1(x,y) = 0 \quad \text{in } \Omega\times Q_1, \quad n\cdot\nabla_y u_1(x,y)\big|_{y\in\Gamma} = -n\cdot\nabla_x u. \tag{6.9}$$

Proof. Let us consider the integral identity (6.1) on the test functions $\phi_\varepsilon(x) = \varepsilon\psi(x)b(\frac{x}{\varepsilon})$ such that $\psi\in C_0^\infty(\Omega)$ and $b(y)\in C^\infty_\#(Q)$; then

$$\varepsilon\int_{\Omega_1^\varepsilon}\nabla u_\varepsilon\cdot\nabla(\psi(x)b(\frac{x}{\varepsilon}))\,dx + \varepsilon^2\int_{\Omega_0^\varepsilon}\nabla u_\varepsilon\cdot\nabla(\psi(x)b(\frac{x}{\varepsilon}))\,dx$$
$$= \varepsilon^2\omega^\varepsilon\int_{\Omega_1^\varepsilon} u_\varepsilon\psi(x)b(\frac{x}{\varepsilon})\,dx + \varepsilon\omega^\varepsilon\int_{\Omega_0^\varepsilon} u_\varepsilon\psi(x)b(\frac{x}{\varepsilon})\,dx. \tag{6.10}$$

Normalization (6.4) shows that the right-hand side of (6.10) tends to zero as $\varepsilon\to 0$. Since

$$\nabla(\psi(x)b(\frac{x}{\varepsilon})) = \varepsilon^{-1}\psi(x)\nabla_y b(y) + b(y)\nabla_x\psi(x), \quad y = \frac{x}{\varepsilon}, \tag{6.11}$$

the second term on the left-hand side of (6.10) becomes

$$\varepsilon\int_{\Omega_0^\varepsilon}\psi(x)\nabla u_\varepsilon\cdot(\nabla_y b)\big|_{y=\frac{x}{\varepsilon}}\,dx + \varepsilon^2\int_{\Omega_0^\varepsilon} b(\frac{x}{\varepsilon})\nabla u_\varepsilon\cdot\nabla\psi(x)\,dx. \tag{6.12}$$

Note that by Lemma 1, the sequence $\varepsilon^{1/2}\chi_0^\varepsilon\nabla u_\varepsilon$ is L^2-bounded. Therefore, up to a subsequence, it is two-scale weakly convergent. Then we can pass to the limit in both terms of (6.12), which become zero.

Finally, we can pass to the limit in the first term of identity (6.10). By Lemma 2, we obtain

$$\varepsilon\int_{\Omega_1^\varepsilon}\nabla u_\varepsilon\cdot\nabla(\psi(x)b(\frac{x}{\varepsilon}))\,dx$$
$$= \int_{\Omega_1^\varepsilon}\psi(x)\nabla u_\varepsilon\cdot(\nabla_y b)\big|_{y=\frac{x}{\varepsilon}}\,dx + \varepsilon\int_{\Omega_1^\varepsilon} b(\frac{x}{\varepsilon})\nabla u_\varepsilon\cdot\nabla\psi(x)\,dx$$
$$\to \int_\Omega\psi(x)\int_Q\chi_1(y)[\nabla u(x) + \nabla_y u_1(x,y)]\nabla_y b(y)\,dy\,dx, \quad \varepsilon\to 0.$$

Since all the other terms in (6.10) vanish in the limit, we have

$$\int_{\Omega} \psi(x) \int_{Q} \chi_1(y)[\nabla u(x) + \nabla_y u_1(x,y)]\nabla_y b(y)\, dy\, dx = 0.$$

Since $\psi \in C_0^\infty(\Omega)$ is an arbitrary function from a set that is dense in $L^2(\Omega)$,

$$\int_{Q_1} [\nabla u(x) + \nabla_y u_1(x,y)]\nabla_y b(y)\, dy = 0, \quad \forall b \in C_{\#}^\infty(Q). \tag{6.13}$$

Thus, the vector $[\nabla u(x) + \nabla_y u_1(x,y)]$ is orthogonal to all potential vectors; therefore, it is solenoidal or divergent-free, i.e., $\operatorname{div}_y[\nabla u(x) + \nabla_y u_1(x,y)] = 0$. The latter obviously shows that $\Delta_y u_1(x,y) = 0$ in $L^2(Q)$. This together with (6.13) reconstruct the boundary condition in (6.9) by means of distributions.

Corollary 1. *Let $N_j(y)$ be a unique solution in $H^1(Q)$ to the problem*

$$\Delta_y N_j(y) = 0 \ \text{ in } Q_1, \quad n \cdot \nabla_y N_j = -n_j \ \text{ on } \Gamma, \quad \int_{Q_1} N_j(y)\, dy = 0, \tag{6.14}$$

where n_j is the jth component of the normal n. Then u_1 from Lemma 3 can be given by

$$u_1(x,y) = N_k(y)\partial_{x_k} u(x), \tag{6.15}$$

where we use the conventional summation over repeating indices.

Let $A^{hom} = (A^{hom}_{jk})_{j,k=1}^n$ be the classical homogenized matrix for periodic perforated domains (see, e.g., [JKO94]),

$$A^{hom}_{jk} = |Q_1|\delta_{jk} + \int_{Q_1} \partial_{y_j} N_k\, dy. \tag{6.16}$$

Lemma 4. *Let $\omega^\varepsilon = \varepsilon^{-1}\lambda^\varepsilon$ tend to ω and $u_\varepsilon \xrightarrow{2} u(x)$ as $\varepsilon \to 0$. Then either $u \equiv 0$ or $u \in H_0^1(\Omega)$ is an eigenfunction corresponding to the eigenvalue ω of the problem*

$$-\operatorname{div} A^{hom} \nabla_x u(x) \ = \ \omega |Q_0| u(x) \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega. \tag{6.17}$$

The spectrum of (6.17) consists of a countable set of eigenvalues of finite multiplicity

$$0 < \omega_1 < \omega_2 \le \cdots \le \omega_j \le \cdots \to +\infty.$$

The corresponding eigenfunctions v_j form an orthonormal basis in $L_2(\Omega)$,

$$\int_{\Omega} u_j u_k\, dx = \delta_{jk}.$$

Proof of Lemma 4. Passing to the limit as $\varepsilon \to 0$ in (6.1) for $\phi(x) \in C_0^\infty(\Omega)$, let us first consider the potential energy form being represented by the first term in (6.1). By Lemma 2,

$$\int_{\Omega_1^\varepsilon} \nabla u_\varepsilon \cdot \nabla \phi \, dx = \int_\Omega \chi_1^\varepsilon \nabla u_\varepsilon \cdot \nabla \phi \, dx \\ \to \int_\Omega \left(|Q_1| \nabla_x u(x) + \int_{Q_1} \nabla_y u_1(x,y) \, dy \right) \cdot \nabla_x \phi(x) \, dx. \quad (6.18)$$

Moreover, (6.15) shows that the right-hand side of (6.18) is equal to

$$\int_\Omega \left(|Q_1| \nabla_x u(x) + \partial_{x_k} u(x) \int_{Q_1} \nabla_y N_k(y) \, dy \right) \cdot \nabla_x \phi(x) \, dx. \quad (6.19)$$

The rest of the potential energy form also possesses a limit since, by Lemma 1, it is the product of a bounded sequence and an infinitely small one ($\varepsilon \to 0$),

$$\varepsilon \int_{\Omega_0^\varepsilon} \nabla u_\varepsilon(x) \cdot \nabla \phi(x) \, dx = \varepsilon^{1/2} \int_\Omega (\varepsilon^{1/2} \chi_0^\varepsilon \nabla u_\varepsilon(x)) \cdot \nabla \phi(x) \, dx \to 0. \quad (6.20)$$

Second, since normalization (6.4) holds, the kinetic energy form, which is the second term in (6.1), also has a limit

$$\lambda^\varepsilon \int_\Omega \rho_\varepsilon(x) u_\varepsilon \phi \, dx = \varepsilon \omega^\varepsilon \int_\Omega \chi_1^\varepsilon u_\varepsilon \phi \, dx + \omega^\varepsilon \int_\Omega \chi_0^\varepsilon u_\varepsilon \phi \, dx \\ \to \omega \int_\Omega \int_Q \chi_0(y) u(x) \phi(x) \, dy \, dx = \omega |Q_0| \int_\Omega u(x) \phi(x) \, dx. \quad (6.21)$$

Combining (6.19)–(6.21), we find that the limit function u, which belongs to $H_0^1(\Omega)$ by Lemma 2, satisfies the variational problem

$$\int_\Omega \left(|Q_1| \nabla_x u(x) + \partial_{x_k} u(x) \int_{Q_1} \nabla_y N_k(y) \, dy \right) \cdot \nabla_x \phi(x) \, dx \\ - \omega |Q_0| \int_\Omega u(x) \phi(x) \, dx = 0 \quad \forall \phi \in C_0^\infty(\Omega), \quad (6.22)$$

which is a weak formulation of (6.17).

Lemma 5. *The first eigenvalue λ_1^ε of (6.2)–(6.3) satisfies the estimate $c\varepsilon \leq \lambda_1^\varepsilon \leq C\varepsilon$ with positive constants c and C independent of ε.*

Proof. By the minimax principle, we have

$$\lambda_1^\varepsilon = \min_{0 \not\equiv v \in H_0^1(\Omega)} \frac{(v,v)_{\mathcal{H}^\varepsilon}}{(v,v)_{\mathcal{L}^\varepsilon}} = \min_{0 \not\equiv v \in H_0^1(\Omega)} \frac{\int_{\Omega_1^\varepsilon} |\nabla v|^2 dx + \varepsilon \int_{\Omega_0^\varepsilon} |\nabla v|^2 dx}{\int_{\Omega_1^\varepsilon} v^2 dx + \varepsilon^{-1} \int_{\Omega_0^\varepsilon} v^2 dx}. \quad (6.23)$$

First, we show the estimate from below. Then, decreasing the numerator and increasing the denominator for $\varepsilon \in (0,1)$, we obtain

$$\lambda_1^\varepsilon \geq \min_{0 \not\equiv v \in H_0^1(\Omega)} \frac{\int_{\Omega_1^\varepsilon} |\nabla v|^2 dx + \varepsilon^2 \int_{\Omega_0^\varepsilon} |\nabla v|^2 dx}{\varepsilon^{-1} \int_\Omega v^2 dx} = \varepsilon \mu_1^\varepsilon, \tag{6.24}$$

where μ_1^ε is the first eigenvalue of the corresponding double porosity model, see [Zh00]. By [Zh00, Th. 8.1], there exists a limit $\mu_1^\varepsilon \to \mu$, where $\mu > 0$ is the bottom of the spectrum of the homogenized operator. Therefore, for ε small enough, $\mu_1^\varepsilon \geq \frac{\mu}{2}$ and, finally, $\lambda_1^\varepsilon \geq \varepsilon \frac{\mu}{2}$.

In the case of the geometric configuration A (see page 54), we refer to [BKS08, Th. 4.6] for the proof that there exist a constant $C_1 > 0$ and eigenvalues λ^ε satisfying

$$|\lambda^\varepsilon - \varepsilon\omega_1| \leq C_1 \varepsilon^{5/4} \tag{6.25}$$

for sufficiently small ε. In the case of the geometric configuration B, the proof of [BKS08, Th. 4.6] can be literally adapted from the above since the absence of inclusions touching or intersecting the boundary does not change the main arguments. Let λ_k^ε be one of the eigenvalues satisfying (6.25), $k \in \mathbb{N}$. Then

$$\lambda_k^\varepsilon \leq \varepsilon\omega_1 + C_1 \varepsilon^{5/4} \leq C\varepsilon.$$

By the counting convention, $\lambda_1^\varepsilon \leq \lambda_k^\varepsilon \leq C\varepsilon$.

Note that [BKS08, Th. 4.6] provides a more general result than the one stated in the proof of Lemma 5. In particular, for arbitrary ω_j and sufficiently small ε there exist $C_k > 0$ and λ^ε satisfying

$$|\varepsilon^{-1}\lambda^\varepsilon - \omega_k| \leq C_k \varepsilon^{1/4}. \tag{6.26}$$

Then for $\varepsilon \to 0$ we can choose a sequence $\varepsilon^{-1}\lambda^\varepsilon$ satisfying (6.26) and thus possessing the limit $\varepsilon^{-1}\lambda^\varepsilon \to \omega_k$. Therefore, according to Lemma 1, a certain corresponding eigenfunction subsequence has a weak two-scale limit $u(x)$. By Lemma 4, the limit u is either zero or an eigenfunction of the homogenized problem (6.17). Thus, we have proved the following assertion.

Lemma 6. *The eigenfunction sequences u_ε, corresponding to the eigenvalues λ^ε of* (6.1) *that satisfy* (6.26) *and such that $\|u_\varepsilon\|_{L_2(\Omega)} = 1$, possess weakly convergent subsequences $u_\varepsilon \overset{L^2(\Omega)}{\rightharpoonup} u_k(x)$ and $u_\varepsilon \overset{2}{\rightharpoonup} u_k(x)$, where in both cases the limit is a function of the slow variable only. The limit u_k is either zero or an eigenfunction of the homogenized problem* (6.17)*, which corresponds to the eigenvalue ω_k.*

The result can be improved by showing that $u \not\equiv 0$. This can be done by employing compensated compactness arguments (see, e.g., [Zh00, Lemma 8.2]). Nevertheless, the known methods that do it require the elimination of the geometric configuration of type A when inclusions are touching or intersecting the boundary. After applying compensated compactness arguments, we additionally obtain a strong eigenfunction convergence. A detailed analysis of this situation will be published elsewhere.

Note that there are also other *high-frequency* accumulation points of the spectra for (6.2)–(6.3) as $\varepsilon \to 0$ (see [BKS08]). The analysis of the correspondent eigenfunction convergence at high frequencies requires some additional assumptions and is beyond the scope of this chapter.

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7

Fredholm Characterization of Wiener–Hopf–Hankel Integral Operators with Piecewise Almost Periodic Symbols

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7.1 Introduction

This chapter is concerned with the Fredholm property of matrix Wiener–Hopf–Hankel operators (cf. [BoCa08], [BoCa], and [LMT92]) of the form

$$W_\Phi \pm H_\Phi : [L^2_+(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^N , \tag{7.1}$$

for $N \times N$ matrix-valued functions Φ with entries in the class of piecewise almost periodic elements (see [BoCa] or [BKS02]), and where W_Φ and H_Φ denote matrix Wiener–Hopf and Hankel operators defined by

$$W_\Phi = r_+\mathcal{F}^{-1}\Phi \cdot \mathcal{F} : [L^2_+(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^N \tag{7.2}$$

$$H_\Phi = r_+\mathcal{F}^{-1}\Phi \cdot \mathcal{F}J : [L^2_+(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^N , \tag{7.3}$$

respectively. We are denoting by $L^2(\mathbb{R})$ and $L^2(\mathbb{R}_+)$ the Banach spaces of complex-valued Lebesgue measurable functions φ, for which $|\varphi|^2$ is integrable on $\mathbb{R}$ and $\mathbb{R}_+$, respectively. Moreover, in (7.1)–(7.3) $L^2_+(\mathbb{R})$ denotes the subspace of $L^2(\mathbb{R})$ formed by all functions supported in the closure of $\mathbb{R}_+ = (0, +\infty)$, the operator r_+ performs the restriction from $L^2(\mathbb{R})$ into $L^2(\mathbb{R}_+)$, $\mathcal{F}$ denotes the Fourier transformation, and J is the reflection operator given by the rule $J\Phi(x) = \widetilde{\Phi}(x) = \Phi(-x)$, $x \in \mathbb{R}$.

We are therefore considering Wiener–Hopf–Hankel type operators with the same Fourier symbol in the Wiener–Hopf and the Hankel components. For matrix symbols in the piecewise almost periodic algebra, we will obtain conditions which characterize the Fredholm property of those operators. This characterization will be based on certain factorizations of matrix functions and on spectral properties of other functions which are built from the original Fourier symbols of the integral operators. The present work generalizes some of the results of [BoCa]. In the next sections we start by presenting several notions and auxiliary results which will allow us to reach the main result in the last section.

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7.2 Almost Periodic Functions

A function α of the form $\alpha(x) := \sum_{j=1}^{n} c_j \exp(i\lambda_j x)$, $x \in \mathbb{R}$, where $\lambda_j \in \mathbb{R}$ and $c_j \in \mathbb{C}$, is called an almost periodic polynomial. If we construct the closure of the set of all almost periodic polynomials by using the supremum norm, we will then obtain the AP class of almost periodic functions.

Theorem 1 (Bohr). *Suppose that* $\varphi \in AP$ *and*

$$\inf_{x\in\mathbb{R}} |\varphi(x)| > 0 \,. \tag{7.4}$$

Then the function $\arg\varphi(x)$ *can be defined so that* $\arg\varphi(x) = \lambda x + \psi(x)$*, where* $\lambda \in \mathbb{R}$ *and* $\psi \in AP$*.*

Definition 1 (Bohr mean motion). *Let* $\varphi \in AP$ *and let the condition (7.4) be satisfied. The* Bohr mean motion *of the function* φ *is defined to be the real number* $k(\varphi) := \lim_{\ell\to\infty} \frac{1}{2\ell} \arg\varphi(x)|_{-\ell}^{\ell}$*.*

Let $e_\lambda(x) := e^{i\lambda x}$, $x \in \mathbb{R}$. The subclasses $AP_+ := \mathrm{alg}_{L^\infty(\mathbb{R})}\{e_\lambda : \lambda \geq 0\}$ and $AP_- := \mathrm{alg}_{L^\infty(\mathbb{R})}\{e_\lambda : \lambda \leq 0\}$ of AP are also of interest. In fact, one of the reasons why the last two algebras are very useful is due to the fact that $AP_\pm = AP \cap H^\infty_\pm(\mathbb{R})$ (cf. [BKS02, Corollary 7.7]).

Proposition 1 (cf., e.g., [BKS02]). *Let* $A \subset (0,\infty)$ *be an unbounded set and let* $\{I_\eta\}_{\eta\in A} := \{(x_\eta, y_\eta)\}_{\eta\in A}$ *be a family of intervals* $I_\eta \subset \mathbb{R}$ *such that* $|I_\eta| = y_\eta - x_\eta \to \infty$ *as* $\eta \to \infty$*. If* $\varphi \in AP$*, then the limit* $M(\varphi) := \lim_{\eta\to\infty} \frac{1}{|I_\eta|} \int_{I_\eta} \varphi(x)dx$ *exists, is finite, and is independent of the particular choice of the family* $\{I_\eta\}$*.*

Definition 2. *Let* $\varphi \in AP$*. The number* $M(\varphi)$ *given by Proposition 1 is called the* Bohr mean value *or simply the* mean value *of* φ*.*

In the matrix case the *mean value* is defined entry-wise.

7.3 Matrix *AP* Factorization

Since our results will be obtained through certain factorizations of the involved matrix functions, we will therefore recall the definitions of right and left AP factorization. In this framework we will denote by $\mathcal{G}X$ the group of all invertible elements from a Banach algebra X.

Definition 3. *A matrix function* $\Phi \in \mathcal{G}AP^{N\times N}$ *is said to admit a* right AP factorization *if it can be represented in the form*

$$\Phi(x) \;=\; \Phi_-(x)\, D(x)\, \Phi_+(x) \tag{7.5}$$

for all $x \in \mathbb{R}$, *with* $\Phi_- \in \mathcal{G}AP_-^{N\times N}$, $\Phi_+ \in \mathcal{G}AP_+^{N\times N}$, *and where* D *is a diagonal matrix of the form* $D(x) = \operatorname{diag}(e^{i\lambda_1 x}, \dots, e^{i\lambda_N x})$, $\lambda_j \in \mathbb{R}$. *The numbers* λ_j *are called the* right AP indices *of the factorization. A right AP factorization with* $D = I_{N\times N}$ *is referred to as a* canonical right AP factorization.

In another way, it is said that a matrix function $\Phi \in \mathcal{G}AP^{N\times N}$ *admits a left AP factorization if instead of (7.5) we have* $\Phi(x) = \Phi_+(x)\, D(x)\, \Phi_-(x)$ *for all* $x \in \mathbb{R}$, *and* $\Phi_\pm$ *and* D *having the same properties as above.*

Remark 1. It is readily seen from the above definition that if an invertible almost periodic matrix function Φ admits a right AP factorization, then $\widetilde{\Phi}$ admits a left AP factorization, and also Φ^{-1} admits a left AP factorization.

The vector containing the right AP indices will be denoted by $k(\Phi)$, i.e., in the above case $k(\Phi) := (\lambda_1, \dots, \lambda_N)$. If we consider the case with equal *right AP indices* ($k(\Phi) = (\lambda_1, \lambda_1, \dots, \lambda_1)$), then the matrix $\mathbf{d}(\Phi) := M(\Phi_-)M(\Phi_+)$ is independent of the particular choice of the right AP factorization (cf., e.g., [BKS02, Proposition 8.4]). In this case the matrix $\mathbf{d}(\Phi)$ is called the *geometric mean of* Φ.

7.4 Semi-Almost Periodic and Piecewise Almost Periodic Functions

Let $C(\dot{\mathbb{R}})$ (with $\dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$) represent the (bounded and) continuous functions φ on the real line for which the two limits $\varphi(-\infty) := \lim_{x\to-\infty} \varphi(x)$, $\varphi(+\infty) := \lim_{x\to+\infty} \varphi(x)$ exist and coincide. The common value of these two limits will be denoted by $\varphi(\infty)$. Furthermore, $C_0(\dot{\mathbb{R}})$ will stand for the functions $\varphi \in C(\dot{\mathbb{R}})$ for which $\varphi(\infty) = 0$.

We denote by $PC := PC(\dot{\mathbb{R}})$ the C^*-algebra of all bounded piecewise continuous functions on $\dot{\mathbb{R}}$, and we also put $C(\overline{\mathbb{R}}) := C(\mathbb{R}) \cap PC$, where $C(\mathbb{R})$ denotes the usual set of continuous functions on the real line. Use will also be made of the C^*-algebra $PC_0 := \{\varphi \in PC : \varphi(\pm\infty) = 0\}$.

We are now in a position to define the C^*-algebra of semi-almost periodic elements.

Definition 4. *The* C^**-algebra* SAP *of all* semi-almost periodic functions *on* $\mathbb{R}$ *is the smallest closed subalgebra of* $L^\infty(\mathbb{R})$ *that contains* AP *and* $C(\overline{\mathbb{R}})$:

$$SAP := \operatorname{alg}_{L^\infty(\mathbb{R})}\{AP, C(\overline{\mathbb{R}})\} .$$

In [Sa77] Sarason proved the following theorem which reveals in a different way the structure of the SAP algebra.

Theorem 2. *Let* $u \in C(\overline{\mathbb{R}})$ *be any function for which* $u(-\infty) = 0$ *and* $u(+\infty) = 1$. *If* $\varphi \in SAP$, *then there exist* $\varphi_\ell, \varphi_r \in AP$ *and* $\varphi_0 \in C_0(\dot{\mathbb{R}})$ *such that* $\varphi = (1-u)\varphi_\ell + u\varphi_r + \varphi_0$. *The functions* φ_ℓ, φ_r *are uniquely determined by* φ, *and independent of the particular choice of* u. *The maps* $\varphi \mapsto \varphi_\ell$ *and* $\varphi \mapsto \varphi_r$ *are* C^**-algebra homomorphisms of* SAP *onto* AP.

Remark 2. The last theorem is also valid in the matrix case.

Let us consider the closed subalgebra of $L^\infty(\mathbb{R})$ generated by all the almost periodic and the piecewise continuous functions. We will denote it by $PAP := \mathrm{alg}_{L^\infty(\mathbb{R})}\{AP, PC\}$. It is readily seen that $SAP \subset PAP$. In the scalar case it was proved that $PAP = SAP + PC_0$. The same situation is also valid in the matrix case considering the decomposition entry-wise. In addition, the next proposition is the matrix version of a known corresponding result for the representation of PAP elements in the scalar case (cf., e.g., [BKS02, Proposition 3.15]).

Proposition 2. *(i) If $\Phi \in PAP^{N\times N}$, then there are uniquely determined matrix-valued functions $\Theta_\ell, \Theta_r \in AP^{N\times N}$, and $\Phi_0 \in PC_0^{N\times N}$ such that*

$$\Phi = (1-u)\Theta_\ell + u\Theta_r + \Phi_0 \ ,$$

where $u \in C(\mathbb{R})$, $0 \le u \le 1$, $u(-\infty) = 0$, and $u(+\infty) = 1$.

(ii) If $\Phi \in \mathcal{G}PAP^{N\times N}$, then there exist matrix-valued functions $\Theta \in \mathcal{G}SAP^{N\times N}$ and $\Xi \in \mathcal{G}PC^{N\times N}$ such that $\Xi(-\infty) = \Xi(+\infty) = I_{N\times N}$ and

$$\Phi = \Theta\, \Xi \, .$$

(iii) The elements Θ_ℓ and Θ_r used in (i) coincide with the local representatives of $\Theta \in \mathcal{G}SAP^{N\times N}$ used in (ii), and their unique existence is ensured by Theorem 2 and Remark 2.

Proof. The proof of proposition (i) follows in the same lines as the proof of the scalar case (cf. [BKS02, Proposition 3.15]), and therefore it is omitted here.

The proof of proposition (ii) requires certain differences when compared to the scalar case, and therefore will be performed here for the reader's convenience. Suppose that $\Phi \in \mathcal{G}PAP^{N\times N}$, and put $\Upsilon := (1-u)\Theta_\ell + u\Theta_r$. Then $\Phi = \Upsilon + \Phi_0$. There is an $M \in (0,\infty)$ such that $|\det \Upsilon(x)|$ is bounded away from zero for $|x| > M$, and therefore we can find an element $\Upsilon_0 \in [C_0(\dot{\mathbb{R}})]^{N\times N}$ such that $\Theta := \Upsilon + \Upsilon_0 \in \mathcal{G}SAP^{N\times N}$. This allows us to rewrite Φ in the form

$$\Phi = \Theta + \Phi_0 - \Upsilon_0 \quad = \quad \Theta[I + \Theta^{-1}(\Phi_0 - \Upsilon_0)] =: \Theta\Xi \ , \tag{7.6}$$

where it is clear that $\Xi = \Theta^{-1}\Phi \in \mathcal{G}PC^{N\times N}$ and $\Xi(-\infty) = \Xi(+\infty) = I_{N\times N}$.

The part (iii) follows immediately from the construction made in (ii).

Remark 3. Due to the item (iii) of Proposition 2, Θ_ℓ and Θ_r are also called the local representatives of Φ at $-\infty$ and $+\infty$, respectively.

7.5 The Besicovitch Space

In this section we introduce notation and results about the Besicovitch space. For the corresponding proofs, the reader may consult [BKS02, Chapter 7] and the references therein (cf., e.g., [BKS02, page 130]). Denote by AP^0 the set of all almost periodic polynomials. The Besicovitch space B^2 is defined as the completion of AP^0 with respect to the norm $\|\varphi\|_{B^2} := \left(\sum_\lambda |\varphi_\lambda|^2\right)^{1/2}$, where $\varphi = \sum_\lambda \varphi_\lambda e_\lambda \in AP^0$. Let $\mathbb{R}_B$ denote the Bohr compactification of $\mathbb{R}$ and $d\mu$ the normalized Haar measure on $\mathbb{R}_B$ (see, e.g., [BKS02, Chapter 7]). It is known that AP can be identified with $C(\mathbb{R}_B)$ and also that we can identify B^2 with $L^2(\mathbb{R}_B, d\mu)$. Thus, B^2 is a (nonseparable) Hilbert space, and the inner product in $B^2 = L^2(\mathbb{R}_B, d\mu)$ is given by

$$(f, g) := \int_{\mathbb{R}_B} f(\xi)\overline{g(\xi)}\, d\mu(\xi). \tag{7.7}$$

For $f, g \in AP$ it also holds that $(f, g) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\overline{g(x)}\, dx$. Since $\mu(\mathbb{R}_B) = 1$ is finite, AP is contained in B^2. Moreover, AP is a dense subset of B^2.

The Cauchy–Schwarz inequality shows that the mean value $M(f) := \int_{\mathbb{R}_B} f(\xi)\, d\mu(\xi)$ exists and is finite for every $f \in B^2$. For $f \in B^2$, the set $\Omega(f) := \{\lambda \in \mathbb{R} : M(fe_{-\lambda}) \neq 0\}$ is called the Bohr–Fourier spectrum of f and can be shown to be at most countable. Taking into account (7.7), one can prove that for every $f \in B^2$, $\|f\|^2_{B^2} = \sum_{\lambda\in\Omega(f)} |M(fe_{-\lambda})|^2$. Let $\ell^2(\mathbb{R})$ denote the collection of all functions $f : \mathbb{R} \to \mathbb{C}$ for which the set $\{\lambda \in \mathbb{R} : f(\lambda) \neq 0\}$ is at most countable, and $||f||^2_{\ell^2(\mathbb{R})} := \sum |f(\lambda)|^2 < \infty$. Further, $\ell^\infty(\mathbb{R})$ is defined as the set of all functions $f : \mathbb{R} \to \mathbb{C}$ such that $||f||_{\ell^\infty(\mathbb{R})} := \sup_{\lambda\in\mathbb{R}} |f(\lambda)| < \infty$. Note that $\ell^2(\mathbb{R})$ is a (nonseparable) Hilbert space with pointwise operations and the inner product $(f, g) := \sum_{\lambda\in\mathbb{R}} f(\lambda)\overline{g(\lambda)}$, and that $\ell^\infty(\mathbb{R})$ is a C^*-algebra with pointwise operation and the norm $|| \cdot ||_{\ell^\infty(\mathbb{R})}$.

The map $F_B : \ell^2(\mathbb{R}) \to B^2$ which sends a function $f \in \ell^2(\mathbb{R})$ with a finite support to the function $(F_B f)(x) = \sum_{\lambda\in\mathbb{R}} f(\lambda)e^{i\lambda x}$ $(x \in \mathbb{R})$ can be extended by continuity to all $\ell^2(\mathbb{R})$. The operator F_B is referred to as the Bohr–Fourier transform. This operator is an isometric isomorphism in the above-mentioned setting, and its inverse acts by the rule

$$F_B^{-1} : B^2 \to \ell^2(\mathbb{R}), \quad (F_B^{-1} f)(\lambda) = M(fe_{-\lambda}), \quad \lambda \in \mathbb{R}\,.$$

If $a \in \ell^\infty(\mathbb{R})$, then the operator $\psi(a) : B^2 \to B^2$ defined by $\psi(a) := F_B\, a \cdot F_B^{-1}$ is bounded.

7.6 Generalized Matrix *AP* Factorization

Let $B^2_\pm$ denote the Hilbert spaces consisting of the functions in B^2 with the Bohr–Fourier spectra in $\mathbb{R}_\pm = \{x \in \mathbb{R} : \pm x \geq 0\}$.

Definition 5. *A* generalized right *AP* factorization *of a matrix function* $\Phi \in \mathcal{G}AP^{N\times N}$ *is a representation*

$$\Phi = \Phi_- D \Phi_+ \ , \tag{7.8}$$

where $D = \operatorname{diag}(e_{\lambda_1}, \ldots, e_{\lambda_N})$ *with* $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$ *and* $\Phi_- \in \mathcal{G}[B_-^2]^{N\times N}$, $\Phi_+ \in \mathcal{G}[B_+^2]^{N\times N}$, $\Phi_- \widetilde{P} \Phi_-^{-1} I \in \mathcal{L}(B_N^2)$. *Here* $\widetilde{P}$ *is the projection* $\widetilde{P} := F_B \chi_+ F_B^{-1} \in \mathcal{L}(B_N^2)$ *(with* χ_+ *being the characteristic function of* $\mathbb{R}_+$*).*

The numbers λ_j *are called the* right AP indices *of the factorization. A generalized right AP factorization with* $D = I_{N\times N}$ *is referred to as a* canonical generalized right AP factorization.

In another way, it is said that a matrix function $\Phi \in \mathcal{G}AP^{N\times N}$ *admits a* generalized left *AP* factorization *if instead of (7.8) we have* $\Phi = \Phi_+ D \Phi_-$ *with* $\Phi_\pm$ *and* D *having the same properties as above.*

If Φ admits a right generalized AP factorization, then $\widetilde{\Phi}$ admits a left generalized AP factorization, and also Φ^{-1} admits a left generalized AP factorization.

The corresponding definition of the geometric mean value is literally the same as in Section 7.3.

7.7 Matrix Wiener–Hopf Operators with *PC* Symbols

We recall here some of the essential facts from the theory of Wiener–Hopf and Hankel operators. The following equality is well known:

$$W_{\Phi\Psi} = W_\Phi \ell_0 W_\Psi + H_\Phi \ell_0 H_{\widetilde{\Psi}} \ , \tag{7.9}$$

for $\Phi, \Psi \in [L^\infty(\mathbb{R})]^{N\times N}$. The next proposition is the matrix version of the classical scalar case, which is also obviously valid for the matrix case (one can derive the matrix case result by using the scalar one entry-wise).

Proposition 3. *If* $\Theta \in [C(\dot{\mathbb{R}})]^{N\times N}$*, then the Hankel operators* H_Θ *and* $H_{\widetilde{\Theta}}$ *are compact.*

We can equivalently rewrite (7.9) as $W_{\Phi\Psi} - W_\Phi \ell_0 W_\Psi = H_\Phi \ell_0 H_{\widetilde{\Psi}}$, and therefore Proposition 3 directly yields the following known result.

Theorem 3. *If* $\Phi, \Psi \in [L^\infty(\mathbb{R})]^{N\times N}$ *and at least one of the functions* Φ, Ψ *belongs to* $[C(\dot{\mathbb{R}})]^{N\times N}$*, then* $W_{\Phi\Psi} - W_\Phi \ell_0 W_\Psi$ *is compact.*

Now, employing a continuous partition of the identity, one can sharpen Theorem 3 as follows.

Theorem 4. *If* $\Phi, \Psi \in PC^{N\times N}$ *and if at each point* $x_0 \in \dot{\mathbb{R}}$ *at least one of the functions* Φ *and* Ψ *is continuous, then* $W_{\Phi\Psi} - W_\Phi \ell_0 W_\Psi$ *is compact.*

Proof. The result can be proved by following the same arguments as in the scalar case [Kr87, Lemma 16.2], with corresponding changes for matrices in the places of functions. Namely, let $x_1, \ldots, x_\ell$ and $x_{\ell+1}, \ldots, x_r$ denote all the points of discontinuity of the matrix functions Φ and Ψ, respectively. Then, let Θ and Ξ be continuous matrix functions on $\dot{\mathbb{R}}$ with the following properties: $\Theta(x_k) = 0_{N\times N},\ k = 1, \ldots, \ell,\ \Xi(x_k) = 0_{N\times N},\ k = \ell+1, \ldots, r$, and $\Theta + \Xi \equiv I_{N\times N}$. This construction of Θ and Ξ make it clear that $\Phi\Theta$ and $\Xi\Psi$ are continuous on $\dot{\mathbb{R}}$. From Theorem 3 and $\Theta + \Xi = I_{N\times N}$, we have

$$\begin{aligned}
W_{\Phi\Psi} &= W_{\Phi(\Theta+\Xi)\Psi} = W_{\Phi\Theta\Psi} + W_{\Phi\Xi\Psi} = W_{\Phi\Theta}\ell_0 W_\Psi + K_1 + W_\Phi \ell_0 W_{\Xi\Psi} + K_2 \\
&= W_{\Phi\Theta}\ell_0 W_\Psi + W_\Phi \ell_0 W_{\Xi\Psi} + K_3 \\
&= (W_\Phi \ell_0 W_\Theta + K_4)\ell_0 W_\Psi + W_\Phi \ell_0 (W_\Xi W_\Psi + K_5) + K_3 \\
&= W_\Phi \ell_0 W_\Theta \ell_0 W_\Psi + K_6 + W_\Phi \ell_0 W_\Xi \ell_0 W_\Psi + K_7 + K_3 \\
&= W_\Phi \ell_0 (W_\Theta + W_\Xi)\ell_0 W_\Psi + K_8 \\
&= W_\Phi \ell_0 W_\Psi + K_8 \ ,
\end{aligned}$$

where K_i are compact operators ($i = 1, \ldots, 8$). From here we derive that $W_{\Phi\Psi} - W_\Phi \ell_0 W_\Psi$ is a compact operator.

Theorem 5 (cf., e.g., [BKS02, Theorem 5.10]). *Let $\Phi \in \mathcal{G}PC^{N\times N}$, and denote by $\mathrm{sp}[\Phi^{-1}(x-0)\Phi(x+0)]$ the set of eigenvalues of the matrix $\Phi^{-1}(x-0)\Phi(x+0)$. In view of W_Φ to have the Fredholm property it is necessary and sufficient that $\mathrm{sp}[\Phi^{-1}(x-0)\Phi(x+0)] \cap (-\infty, 0] = \emptyset$, for all $x \in \dot{\mathbb{R}}$.*

7.8 Matrix Wiener–Hopf Operators with *SAP* Symbols

Regarding matrix Wiener–Hopf operators with SAP symbols, a Fredholm characterization of this kind of operators is now well known.

Theorem 6 ([BKS02, Theorem 18.18]). *Let $\Phi \in SAP^{N\times N}$. The operator W_Φ is Fredholm if and only if the following three conditions are satisfied:*

(i) $\Phi \in \mathcal{G}SAP^{N\times N}$,
(ii) W_{Φ_ℓ} *and* W_{Φ_r} *are invertible operators,*
(iii) $\mathrm{sp}[\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)] \cap (-\infty, 0] = \emptyset$, *where* $\mathrm{sp}[\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)]$ *stands for the set of the eigenvalues of the matrix* $\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell) := [\mathbf{d}(\Phi_r)]^{-1}\mathbf{d}(\Phi_\ell)$.

7.9 Matrix Wiener–Hopf Operators with *PAP* Symbols

The next proposition is the matrix version of a known corresponding result for the scalar case (cf., e.g., [BKS02, Proposition 3.15]).

Proposition 4. *If $\Phi \in \mathcal{G}PAP^{N\times N}$, then there exist matrix-valued functions $\Theta \in \mathcal{G}SAP^{N\times N}$ and $\Xi \in \mathcal{G}PC^{N\times N}$ such that $\Xi(-\infty) = \Xi(+\infty) = I_{N\times N}$,*

$$\Phi = \Theta\, \Xi\,, \tag{7.10}$$

and

$$W_\Phi = W_\Theta \ell_0 W_\Xi + K_1 = W_\Xi \ell_0 W_\Theta + K_2 \tag{7.11}$$

with compact operators K_1, K_2.

Proof. The fact that the factorization (7.10) is always possible under the conditions of the present theorem was deduced in the proof of Proposition 2. Hence, let us assume that Φ is factorized and is given by the formula (7.10). Since Θ is continuous on $\mathbb{R}$ and Ξ is continuous at ∞, we have that Θ and Ξ do not have common points of discontinuity. Now reasoning in a similar way as in the proof of Theorem 4 (e.g., considering two continuous matrix functions on $\dot{\mathbb{R}}$, such that the sum of them is the identity matrix, and vanishing at the points of discontinuity of Θ and Ξ) and also taking profit from Theorem 3, we deduce that (7.11) holds for compact operators K_1 and K_2.

The next result is only the matricial formulation of the corresponding scalar case in which the known scalar arguments also turn out to be valid in the more general matricial case. Anyway, we will present here its complete proof for the reader's convenience.

Theorem 7. *Let $\Phi \in PAP^{N\times N}$. If $\Phi \notin \mathcal{G}PAP^{N\times N}$, then W_Φ is not semi-Fredholm. Assume now that $\Phi \in \mathcal{G}PAP^{N\times N}$, then W_Φ is Fredholm if and only if*

(i) Φ_ℓ *and* Φ_r *admit a canonical generalized right AP factorization,*
(ii) $\mathrm{sp}[\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)] \cap (-\infty, 0] = \emptyset$,
(iii) $\mathrm{sp}[\Phi^{-1}(x-0)\Phi(x+0)] \cap (-\infty, 0] = \emptyset$,

for all $x \in \mathbb{R}$.

Proof. If $\Phi \notin \mathcal{G}PAP^{N\times N}$, then $\Phi \notin \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$ and therefore W_Φ is not semi-Fredholm due to the corresponding Simonenko result [Si68].

Let us now consider $\Phi \in \mathcal{G}PAP^{N\times N}$. Then we can write (cf. formula (7.10)) $\Phi = \Theta\Xi$ (with $\Theta \in \mathcal{G}SAP^{N\times N}$, $\Xi \in \mathcal{G}PC^{N\times N}$, and $\Xi(\pm\infty) = I_{N\times N}$) such that $W_\Phi = W_\Theta \ell_0 W_\Xi + K$, for a compact operator K. From here we infer that W_Φ is a Fredholm operator if and only if W_Θ and W_Ξ are also Fredholm operators. In the present context, these last two operators are Fredholm if and only if the conditions of the theorem are satisfied. More precisely, since W_Θ is a Wiener–Hopf operator with an invertible semi-almost periodic matrix symbol, and with lateral representatives $\Theta_\ell = \Phi_\ell$ and $\Theta_r = \Phi_r$ (cf. Proposition 2), then W_Θ is Fredholm if and only if (cf. Theorem 6) Φ_ℓ and Φ_r admit a canonical generalized right AP factorization, and $\mathrm{sp}[\mathbf{d}^{-1}(\Theta_r)\mathbf{d}(\Theta_\ell)] \cap (-\infty, 0] = \emptyset$.

We turn now to the operator W_Ξ. This operator has an invertible piecewise continuous matrix symbol. Therefore, applying Theorem 5, we obtain that W_Ξ is Fredholm if and only if $\mathrm{sp}[\Xi^{-1}(x-0)\Xi(x+0)] \cap (-\infty, 0] = \emptyset$, $x \in \mathbb{R}$. Now we simply have to observe that $\Xi^{-1}(x-0)\Xi(x+0) = \Phi^{-1}(x-0)\Phi(x+0)$, to reach the final conclusion (recall also that ℓ_0 is an invertible operator).

7.10 Matrix Wiener–Hopf–Hankel Operators with PAP Symbols

We are now in a position to present the main theorem of this chapter.

Theorem 8. *Let $\Phi \in \mathcal{G}PAP^{N\times N}$. Then $W_\Phi + H_\Phi$ and $W_\Phi - H_\Phi$ are both Fredholm operators if and only if*

(i) $\Phi_\ell \widetilde{\Phi_r^{-1}}$ *admits a canonical generalized right AP factorization,*
(ii) $\mathrm{sp}[\mathbf{d}(\Phi_\ell \widetilde{\Phi_r^{-1}})] \cap i\mathbb{R} = \emptyset$,
(iii) $\mathrm{sp}[\Phi(-x+0)\Phi^{-1}(x-0)\Phi(x+0)\Phi^{-1}(-x-0)] \cap (-\infty, 0] = \emptyset$, $\quad x \in \mathbb{R}$.

Proof. Part of the proof of this main theorem is based on what is called the equivalence after extension operator relation (cf., e.g., [BaTs92]). Using the Gohberg–Krupnik–Litvinchuk identity (cf., e.g., [KaSa01]), and the methods presented in [CaSp98] we can ensure that $\mathrm{diag}(W_\Phi + H_\Phi, W_\Phi - H_\Phi)$ is equivalent after extension to $W_{\Phi\widetilde{\Phi}^{-1}}$.

We will first prove the "if" part of the theorem. Set $\Psi := \Phi\widetilde{\Phi^{-1}}$ to simplify the notation. Direct computations lead to $\Psi_\ell = \Phi_\ell \widetilde{\Phi_r^{-1}}$ and $\Psi_r = \Phi_r \widetilde{\Phi_\ell^{-1}}$. Therefore, we also have $\Psi_\ell = \widetilde{\Psi_r^{-1}}$. From the hypothesis of the theorem (cf. condition (i) of the present theorem) we have that Ψ_ℓ admits a canonical generalized right AP factorization. Using formula $\Psi_\ell = \widetilde{\Psi_r^{-1}}$, we deduce that Ψ_r also admits a canonical generalized right AP factorization. From now on we will use the notation $\Lambda := \mathbf{d}(\Psi_\ell)$. From condition (ii) of the present theorem we derive that $\mathrm{sp}[\Lambda^2] \cap (-\infty, 0] = \emptyset$. In fact, as far as we know that Ψ_ℓ admits a canonical generalized right AP factorization, we can write it in the normalized way $\Psi_\ell = \Pi_- \Lambda \Pi_+$, where $\Pi_\pm$ have the same factorization properties as the original lateral factors of the canonical generalized factorization but with $M(\Pi_\pm) = I$. Thus, the identity $\Psi_\ell = \Pi_- \Lambda \Pi_+$ allows $\Psi_r = \widetilde{\Psi_\ell^{-1}} = \widetilde{\Pi_+^{-1}} \Lambda^{-1} \widetilde{\Pi_-^{-1}}$, which in particular shows that $\mathbf{d}(\Psi_r) = \Lambda^{-1}$, and hence $\mathbf{d}^{-1}(\Psi_r) = \Lambda$. Consequently, $\Lambda^2 = \mathbf{d}^{-1}(\Psi_r)\mathbf{d}(\Psi_\ell)$ and condition (ii) of the present theorem is equivalent to $\mathrm{sp}[\mathbf{d}^{-1}(\Psi_r)\mathbf{d}(\Psi_\ell)] \cap (-\infty, 0] = \emptyset$.

Condition (iii) allows us to conclude that $\mathrm{sp}[\Psi^{-1}(x-0)\Psi(x+0)] \cap (-\infty, 0] = \emptyset$. Altogether, we can conclude from Theorem 7 that W_Ψ is a Fredholm operator. Employing the above-mentioned equivalence after extension relation, we obtain that $W_\Phi + H_\Phi$ and $W_\Phi - H_\Phi$ are Fredholm operators. Thus the "if" part is proved.

Now we will proceed to prove the "only if" part. Assume that $\Phi \in \mathcal{G}PAP^{N\times N}$ and that $W_\Phi \pm H_\Phi$ have the Fredholm property. Thus, from the above-mentioned equivalence after extension relation, it follows that W_Ψ is also a Fredholm operator. Therefore, condition (i) of Theorem 7 ensures that $\Phi_\ell \widetilde{\Phi_r^{-1}} = \Psi_\ell$ admits a canonical generalized right AP factorization (and also that Ψ_r admits a canonical generalized right AP factorization).

Moreover, the corresponding conditions (ii)–(iii) of Theorem 7 are also satisfied for the function Ψ. As a consequence, reasoning in a very similar way as in the "if" part, we reach the conclusion that $\mathrm{sp}[\mathbf{d}(\Phi_\ell \widetilde{\Phi_r^{-1}})] \cap i\mathbb{R} = \emptyset$, and $\mathrm{sp}[\Phi(-x+0)\Phi^{-1}(x-0)\Phi(x+0)\Phi^{-1}(-x-0)] \cap (-\infty, 0] = \emptyset$. Hence the "only if" part is proved.

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8

Fractal Relaxed Problems in Elasticity

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8.1 Introduction

Many heterogeneous structural materials have a complex or irregular geometry which is not easy to model using classical geometry. Fractal geometry gives a way to model irregularities in a wide range of scientific and engineering domains.

In this chapter, we are interested in the relaxation of some perturbed elastic problems, where the perturbations are localized along fractal zones.

First, we consider an elastic material with thin inclusions of higher rigidity repeated in a self-similar way. We prove that the relaxed elastic energy of the heterogeneous material filling in a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, turns out to be of the form

$$\int_{\Omega} \sigma(u) : e(u)\, dx + c 2^{n-1} \pi \frac{\mu(\chi+1)}{\chi} \frac{1}{\mathcal{H}^d(K)} \int_{K \cap \Omega} |u|^2\, d\mathcal{H}^d,$$

where $\sigma(u)$ is the stress tensor, $e(u)$ is the deformation tensor for some admissible displacement u, c is a positive constant, μ and χ are material coefficients, $\mathcal{H}^d$ is the d-dimensional Hausdorff measure, and d is the similarity dimension of the fractal K. Here $\sigma(u) : e(u)$ denotes the product $\sigma_{ij}(u)\, e_{ij}(u)$, where the summation convention with respect to repeated indices is used.

The relaxation of the scalar version of this problem has been given in [BrNo93], studying the asymptotic behavior of the capacity of sets consisting of thin inclusions.

As a second example, we consider an interfacial problem, namely a fractal defect in a two-dimensional material. We consider a defect Σ associated to a von Koch curve located in a domain Ω which is filled in with an elastic material. A perfect contact is supposed to occur on thin patches disposed on the defect. The relaxed elastic energy is proved to be

$$\int_{\Omega \setminus \Sigma} \sigma(u) : e(u)\, dx + c \frac{\mu}{(\chi+1)} \frac{1}{\mathcal{H}^d(\Sigma)} \int_{\Sigma} |[u]_{\Sigma}|^2\, d\mathcal{H}^d,$$

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where $[u]_\Sigma$ is the jump of the displacement u across Σ.

A complete characterization of contact problems on a fractal interface Σ has been given in [ElBr08]. A typical extra term which appears in the relaxed energies is of the form $\int_\Sigma a_{ij} [u_i]_\Sigma [u_j]_\Sigma \, d\mathcal{H}^d$, where $(a_{ij})_{i,j=1,\cdots,n}$ is a symmetric and positive definite matrix of Borel functions from Σ to $[0,+\infty]$ (see [ElBr08] for more details).

In these two problems, the extra term is generated by the presence of boundary layers near the perturbed zones. The characterization of the asymptotic energy is given using Γ-convergence methods (see [At84], [Da93]).

8.2 Self-Similar Highly Rigid Inclusions

Let Ω be an open and bounded subset of $\mathbb{R}^n$, $n = 2, 3$, with Lipschitz continuous boundary $\partial\Omega$. Denote by $\psi_1, \ldots, \psi_N$ a finite family of contractive similitudes on $\mathbb{R}^n$ with ratio $\rho < 1$. There exists a unique compact subset $K \subset \mathbb{R}^n$ such that $K = \cup_{i=1}^N \psi_i (K)$.

The real number $d = -\ln(N) / \ln(\rho)$ is the dimension of K. For the definitions of the self-similar fractal K, its dimension, and the d-dimensional Hausdorff measure $\mathcal{H}^d$, we refer to [Hu81]. We suppose that the family $(\psi_i)_{i=1,\ldots,N}$ satisfies the open set condition, which requires the existence of a bounded open set $U \subset \mathbb{R}^n$ such that

$$\begin{cases} \mathcal{H}^d (K \backslash U) = 0, \\ \psi_i (U) \subset U \quad \forall i \in \{1, \ldots, N\}, \\ \psi_i (U) \cap \psi_j (U) = \varnothing \quad \text{if } i \neq j. \end{cases}$$

Choose $x_0 \in U$ and define $r = dist\,(x_0, \partial U)/2$. Let $c > 0$. For every $h \in \mathbb{N}$, we set $\varepsilon_h := r\rho^h$ and

$$r_h := \begin{cases} c\,(\varepsilon_h)^d & \text{if } n = 3, \\ \exp\left(\frac{-1}{c}\,(\varepsilon_h)^d\right) & \text{if } n = 2. \end{cases}$$

Let $B(x, R)$ be the ball of radius R and centered at 0, and $T = B(0, 1)$ be the unit ball. We define

$$\begin{cases} x_{i_1,\ldots,i_h} = \psi_{i_1} \circ \cdots \circ \psi_{i_h} (x_0) \qquad i_1, \ldots, i_h \in \{1, 2, \ldots, N\}, \\ T_{i_1,\ldots,i_h} = x_{i_1,\ldots,i_h} + r_h T, \\ T_h = \bigcup_{i_1,\ldots,i_h \in \{1,2,\ldots,N\}} T_{i_1,\ldots,i_h}. \end{cases}$$

We define the space W_h as

$$W_h = \left\{ u \in H^1 (\Omega \backslash T_h; \mathbb{R}^n) \mid u = 0 \text{ on } \partial (\Omega \backslash T_h) \right\}$$

and the functional F_h defined on $L^2 (\Omega; \mathbb{R}^n)$ through

$$F_h(u) = \begin{cases} \int_{\Omega \setminus T_h} \sigma_{ij}(u)\, e_{ij}(u)\, dx & \text{if } u \in W_h, \\ +\infty & \text{otherwise,} \end{cases}$$

where the stress tensor $\sigma(v) = (\sigma_{ij}(v))_{i,j=1,\cdots,n}$ is linked to the linearized deformation tensor $e(v) = (e_{ij}(v))_{i,j=1,\ldots,n}$, $e_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$, through Hooke's law $\sigma_{ij}(v) = \lambda e_{kk}(v)\,\delta_{ij} + 2\mu e_{ij}(v)$, where the summation convention with respect to repeated indices has been used. The constants $\lambda \geq 0$, $\mu > 0$ are the Lamé coefficients of the elastic material. Given $f \in L^2(\Omega;\mathbb{R}^n)$, we consider the following problem:

$$\min_{u \in L^2(\Omega;\mathbb{R}^n)} \left\{ F_h(u) - 2\int_\Omega f \cdot u dx \right\}. \tag{8.1}$$

8.2.1 Local Problems

We consider the following boundary value problems ($m = 1, \ldots, n$):

$$\begin{cases} -\sigma_{ij,j}(w^{n,m}) &= 0 & \text{in } \mathbb{R}^n \setminus \overline{T},\ i = 1, \ldots, n, \\ w^{n,m} &= e_m & \text{on } \partial T, \\ w^{n,m} &\to 0 & \text{as } |y| \to \infty, \text{ for } n = 3, \\ w_i^{n,m} &= \delta_{im} \ln(|y|) + O(1) & \text{as } |y| \to \infty, \text{ for } n = 2, \end{cases} \tag{8.2}$$

where $e_m = (\delta_{1m}, \ldots, \delta_{nm})$ and $\delta_{lm} = 1$ if $m = l$, $\delta_{lm} = 0$ if $m \neq l$. The solution $w^{n,m}$ of this problem can be expressed in terms of the single-layer potential as

$$w_i^{n,m}(x) = -\int_{\partial T} G_{ik}^n(x,.)\, \sigma_{kj}(w^{n,m})\, \nu_j ds_y + c_i \delta_{n2},\ i = 1, \ldots, n,$$

where ν is the outward unit normal with respect to ∂T, c_i, $i = 1, 2$, is some constant (introduced if $n = 2$), and the tensor G^n, $n = 2, 3$, is given through

$$\begin{cases} G^3(x,y) &= \frac{1}{4\pi\mu(\chi+1)}\left(\frac{\chi Id_{\mathbb{R}^3}}{|x-y|} + \frac{(x-y)(x-y)^t}{|x-y|^3}\right), \\ G^2(x,y) &= \frac{1}{2\pi\mu(\chi+1)} \\ & \quad \times \begin{pmatrix} \chi \ln|x-y| - \frac{(x_1-y_1)^2}{|x-y|^2} & -\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^2} \\ -\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^2} & \chi \ln|x-y| - \frac{(x_2-y_2)^2}{|x-y|^2} \end{pmatrix}, \end{cases}$$

where $\chi = \frac{\lambda+3\mu}{\lambda+\mu}$ is Muskhelishvili's parameter and $Id_{\mathbb{R}^3}$ is the 3×3 identity matrix. G^3 is the Kelvin–Somigliana tensor and G^2 is the Boussinesq tensor [PaPe84]. The boundary conditions in (8.2) lead to the following equality:

$$\int_{\partial T} \sigma_{ij}(w^{n,m})\, \nu_j ds_y = -\delta_{im} 2^{n-1} \pi \frac{\mu(\chi+1)}{\chi},$$

from which we deduce the following result.

Lemma 1. *1.* $\int_{\mathbb{R}^3\setminus\overline{T}} \sigma_{ij}\left(w^{3,m}\right) e_{ij}\left(w^{3,l}\right) dx = \delta_{lm} 4\pi\mu\left(\chi+1\right)/\chi$.
2. $\lim_{R\to\infty} \frac{1}{\ln(R)} \int_{B(0,R)\setminus\overline{T}} \sigma_{ij}\left(w^{2,m}\right) e_{ij}\left(w^{2,l}\right) dx = \delta_{lm} 2\pi\mu\left(\chi+1\right)/\chi$.

We now build the local functions $w_h^{n,m}$ through

$$\begin{cases} w_h^{2,m}(x) &= \frac{-1}{\ln(r_h)} \begin{pmatrix} w_h^{2,m}\left(\frac{x-x_{i_1,\dots,i_h}}{r_h}\right) \\ -e_m \end{pmatrix} & \forall i_1,\dots,i_h \in \{1,\cdots,N\}, \\ w_h^{3,m}(x) &= w_h^{3,m}\left(\frac{x-x_{i_1,\dots,i_h}}{r_h}\right) - e_m & \forall i_1,\dots,i_h \in \{1,\dots,N\}. \end{cases}$$

Choose a sequence $(s_h)_h$ of positive numbers satisfying

$$\lim_{h\to\infty} s_h = 0,\ \lim_{h\to\infty} \frac{s_h}{r_h} = \lim_{h\to\infty} \frac{\varepsilon_h}{s_h} = 0.$$

We define the set $B_h(s_h) = \cup_{i_1,\dots,i_h\in\{1,2,\dots,N\}} B\left(x_{i_1,\dots,i_h}, s_h\right)$.

Lemma 2. *For every $\varphi \in C^1(\Omega)$, we have*

$$\begin{aligned} &\lim_{h\to\infty} \int_{\left(B_h(s_h)\setminus\overline{T}_h\right)\cap\Omega} \sigma_{ij}\left(w_h^{n,m}\right) e_{ij}\left(w_h^{n,l}\right) \varphi dx \\ &\quad = c r^d \delta_{im} 2^{n-1}\pi \frac{\mu(\chi+1)}{\chi\mathcal{H}^d(K)} \int_{K\cap\Omega} \varphi(x)\, d\mathcal{H}^d. \end{aligned}$$

Proof. We give the proof for $n=2$, the case $n=3$ following in a similar way. Observe that

$$\begin{aligned} &\int_{\left(B_h(s_h)\setminus\overline{T}_h\right)\cap\Omega} \sigma_{ij}\left(w_h^{2,m}\right) e_{ij}\left(w_h^{2,l}\right) \varphi dx \\ &= \sum_{\substack{i_1,\dots,i_h\in\{1,\dots,N\} \\ B\left(x_{i_1,\dots,i_h}, r_h\right)\subset\Omega}} \frac{-1}{\ln(r_h)} \varphi\left(x_{i_1,\dots,i_h}\right) \\ &\qquad \times \left(\frac{-1}{\ln(r_h)} \int_{\left(B_h(s_h/r_h)\setminus\overline{B}(0,1)\right)} \sigma_{ij}\left(w^{2,m}\right) e_{ij}\left(w^{2,l}\right) dy\right) + o\left(\frac{1}{h}\right), \end{aligned}$$

where $y = \left(x - x_{i_1,\cdots,i_h}\right)/r_h$. Using Lemma 1, we have

$$\begin{aligned} &\lim_{h\to\infty} \int_{\left(B_h(s_h)\setminus\overline{T}_h\right)\cap\Omega} \sigma_{ij}\left(w_h^{n,m}\right) e_{ij}\left(w_h^{n,l}\right) \varphi dx \\ &\quad = 2\pi \frac{\mu(\chi+1)}{\chi} \lim_{h\to\infty} \sum_{\substack{i_1,\dots,i_h\in\{1,\dots,N\} \\ B\left(x_{i_1,\dots,i_h}, r_h\right)\subset\Omega}} \frac{-1}{\ln(r_h)} \varphi\left(x_{i_1,\dots,i_h}\right). \end{aligned}$$

Because $-1/\ln(r_h) = c\left(\varepsilon_h\right)^d = c r^d \rho^{dh} = c r^d / N^h$, one has, according to the ergodicity result [Fa97, Theorem 6.1],

$$\begin{aligned} &\lim_{h\to\infty} \sum_{\substack{i_1,\dots,i_h\in\{1,2,\dots,N\} \\ B\left(x_{i_1,\dots,i_h}, r_h\right)\subset\Omega}} \frac{-1}{\ln(r_h)} \varphi\left(x_{i_1,\dots,i_h}\right) \\ &= c r^d \lim_{h\to\infty} \sum_{\substack{i_1,\dots,i_h\in\{1,\dots,N\} \\ B\left(x_{i_1,\dots,i_h}, r_h\right)\subset\Omega}} \frac{1}{N^h} \varphi\left(x_{i_1,\dots,i_h}\right) = c r^d \frac{1}{\mathcal{H}^d(K)} \int_{K\cap\Omega} \varphi(x)\, d\mathcal{H}^d, \end{aligned}$$

which gives the result.

8.2.2 Convergence

Let u^h be the solution of (8.1). Then $F_h\left(u^h\right) - 2\int_\Omega f \cdot u^h dx \le F_h(0) = 0$. This implies

$$\int_{\Omega\backslash T_h} \sigma_{ij}\left(u^h\right) e_{ij}\left(u^h\right) dx \le C\left(\int_\Omega \left|u^h\right|^2 dx\right)^{1/2}. \tag{8.3}$$

There exists an extension operator P_h from W_h to $L^2\left(\Omega;\mathbb{R}^2\right)$ such that

$$\begin{cases} P_h\left(u^h\right) = \begin{cases} u^h & \text{in } \Omega\backslash T_h, \\ 0 & \text{on } \partial\left(\Omega\backslash T_h\right), \end{cases} \\ \int_\Omega \sigma_{ij}\left(P_h\left(u^h\right)\right) e_{ij}\left(P_h\left(u^h\right)\right) dx \le C\int_{\Omega\backslash T_h} \sigma_{ij}\left(u^h\right) e_{ij}\left(u^h\right) dx, \end{cases}$$

where C is a constant independent of h. Thus, from (8.3),

$$\int_\Omega \sigma_{ij}\left(P_h\left(u^h\right)\right) e_{ij}\left(P_h\left(u^h\right)\right) dx \le C\left\|P_h\left(u^h\right)\right\|_{L^2(\Omega;\mathbb{R}^2)}. \tag{8.4}$$

On the other hand, according to Korn's inequality, there exists a constant C independent of h such that

$$\int_\Omega \left|\nabla\left(P_h\left(u^h\right)\right)\right|^2 dx \le C\int_\Omega \sigma_{ij}\left(P_h\left(u^h\right)\right) e_{ij}\left(P_h\left(u^h\right)\right) dx \tag{8.5}$$

and, from Poincaré's inequality,

$$\int_\Omega \left|P_h\left(u^h\right)\right|^2 dx \le C\int_\Omega \left|\nabla\left(P_h\left(u^h\right)\right)\right|^2 dx. \tag{8.6}$$

From (8.4), (8.5), and (8.6), we deduce that the sequence $\left(P_h\left(u^h\right)\right)_h$ is bounded in $H_0^1\left(\Omega;\mathbb{R}^2\right)$. Thus, up to some subsequence, $\left(P_h u^h\right)_h$ converges to some u in $L^2\left(\Omega;\mathbb{R}^2\right)$-strong. Our main result in this section reads as follows.

Theorem 1. *The sequence $(F_h)_h$ Γ-converges to the functional F_∞ defined through*

$$F_\infty(u) = \begin{cases} \int_\Omega \sigma_{ij}(u) e_{ij}(u) dx + cr^d 2^{n-1}\pi \frac{\mu(\chi+1)}{\chi\mathcal{H}^d(K)} \int_{K\cap\Omega} |u|^2 d\mathcal{H}^d \\ \qquad \text{if } u \in H_0^1\left(\Omega;\mathbb{R}^2\right) \cap L_d^2\left(\Omega\cap K;\mathbb{R}^2\right), \\ +\infty \qquad \text{otherwise,} \end{cases}$$

where the Γ-convergence is taken with respect to the strong topology of $L^2\left(\Omega;\mathbb{R}^2\right)$ and where $L_d^2\left(\Omega\cap K;\mathbb{R}^2\right)$ is the space defined as

$$L_d^2\left(\Omega\cap K;\mathbb{R}^2\right) = \left\{u : \Omega \to \mathbb{R}^2 \mid \int_{K\cap\Omega} |u|^2 d\mathcal{H}^d < \infty\right\}.$$

Proof. Let $\varphi_{i_1,\ldots,i_h}$ be a smooth truncation function satisfying

$$\varphi_{i_1,\ldots,i_h}(x) = \begin{cases} 1 & \text{in } B\left(x_{i_1,\ldots,i_h}, \frac{s_h}{2}\right) \quad i_1,\ldots,i_h \in \{1,\ldots,N\}, \\ 0 & \text{in } \Omega \backslash B\left(x_{i_1,\ldots,i_h}, s_h\right) \quad i_1,\ldots,i_h \in \{1,\ldots,N\}. \end{cases}$$

For every $u \in C_c^1\left(\Omega;\mathbb{R}^2\right)$, we define the test function u_0^h through

$$u_0^h(x) = u\left(1 - \varphi_{i_1,\ldots,i_N}\right) + \varphi_{i_1,\ldots,i_N} w_h^{n,m} u_m. \tag{8.7}$$

Then $u_0^h \in W_h$ and $\left(u_0^h\right)_h$ converges to u in $L^2\left(\Omega;\mathbb{R}^2\right)$-strong. Using Lemma 2, one can verify that $\lim_{h\to\infty} F_h\left(u_0^h\right) = F_\infty(u)$.

Let u be any function in $H_0^1\left(\Omega;\mathbb{R}^2\right) \cap L_d^2\left(\Omega\cap K;\mathbb{R}^2\right)$. There exists a sequence $\left(u^k\right)_k \subset C_c^1\left(\Omega;\mathbb{R}^2\right)$ converging to u in the strong topology of $H_0^1\left(\Omega;\mathbb{R}^2\right)$. Defining the sequence $\left(u_0^{k,h}\right)_h$ through (8.7) for every k, one can see that $\left(u_0^{k,h}\right)_h$ converges to u^k in $L^2\left(\Omega;\mathbb{R}^2\right)$-strong, and $\lim_{h\to\infty} F_h\left(u_0^{k,h}\right) = F_\infty\left(u^k\right)$. The continuity of F_∞ with respect to the strong topology of $H_0^1\left(\Omega;\mathbb{R}^2\right)$ implies that $\lim_{k\to\infty}\lim_{h\to\infty} F_h\left(u_0^{k,h}\right) = F_\infty(u)$. Then, we conclude using the diagonalization argument of [At84, Corollary 1.18].

Let now $\left(u^h\right)_h$ be any sequence such that $u^h \in W_h$ and $\left(u^h\right)_h$ converges to $u \in H_0^1\left(\Omega;\mathbb{R}^2\right) \cap L_d^2\left(\Omega\cap K;\mathbb{R}^2\right)$ in the strong topology of $L^2\left(\Omega;\mathbb{R}^2\right)$. We write the following subdifferential inequality:

$$F_h\left(u^h\right) \geq F_h\left(u_0^{k,h}\right) + 2\int_{\Omega\backslash T_h} \sigma_{ij}\left(u_0^{k,h}\right) e_{ij}\left(u^h - u_0^{k,h}\right) dx.$$

We observe that

$$\int_{\Omega\backslash T_h} \sigma_{ij}\left(u_0^{k,h}\right) e_{ij}\left(u^h - u_0^{k,h}\right) dx = -\int_{\Omega\backslash T_h} \sigma_{ij,j}\left(u_0^{k,h}\right)\left(u_i^h - \left(u_0^{k,h}\right)_i\right) dx.$$

Because $\sigma_{ij,j}\left(w_h^{2,m}\right) = 0$, one gets

$$\lim_{h\to\infty} \int_{\Omega\backslash T_h} \sigma_{ij}\left(u_0^{k,h}\right) e_{ij}\left(u^h - u_0^{k,h}\right) dx = -\int_\Omega \sigma_{ij,j}\left(u^k\right)\left(u_i - \left(u_0^k\right)_i\right) dx.$$

Thus

$$\liminf_{h\to\infty} F_h\left(u^h\right) \geq F_\infty\left(u^k\right) - 2\int_\Omega \sigma_{ij,j}\left(u^k\right)\left(u_i - \left(u_0^k\right)_i\right) dx.$$

Letting k go to ∞, one gets $\liminf_{h\to\infty} F_h\left(u^h\right) \geq F_\infty(u)$, which ends the proof.

From the properties of the Γ-convergence, we deduce the following convergence result.

Corollary 1. *The sequence* $\left(u^h\right)_h$*, where* u^h *is the solution of (8.1), converges in the strong topology of* $L^2\left(\Omega;\mathbb{R}^2\right)$ *to the solution* $u \in H_0^1\left(\Omega;\mathbb{R}^2\right) \cap L_d^2\left(\Omega \cap K;\mathbb{R}^2\right)$ *of the limit minimization problem*

$$\min_{u\in L^2(\Omega;\mathbb{R}^2)} \left\{F_\infty(u) - 2\int_\Omega f\cdot u dx\right\},$$

and $\lim_{h\to\infty} F_h\left(u^h\right) = F_\infty(u)$.

8.3 Interface Case: Fractal Defect

We define the contractive similitudes ψ_1, ψ_2, ψ_3, and ψ_4 on $\mathbb{R}^2$ as $\psi_k(x) = a_k + R_k x$, with

$$\begin{cases} a_1 = (0,0), & R_1 = Id_{\mathbb{R}^2}, \\ a_2 = (1/3,0), & R_2 = \begin{pmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{pmatrix}, \\ a_3 = (2/3,0), & R_3 = \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}, \\ a_4 = (1,0), & R_4 = Id_{\mathbb{R}^2}. \end{cases}$$

The compact set Σ defined as $\Sigma = \cup_{i=1}^4 \psi_i(\Sigma)$ is the von Koch curve of Hausdorff dimension $d = \ln(4)/\ln(3)$. We consider a bounded domain Ω of $\mathbb{R}^2$, with Lipschitz continuous boundary $\partial\Omega$, such that $\Sigma \subset \Omega$. The point $x_0 = \left(1/2, \sqrt{3}/2\right) = a_2 + R_2 a_2$ is the summit of Σ. We define, for every $h \in \mathbb{N}$,

$$\begin{cases} x_{i_1,\dots,i_h} = \psi_{i_1}\circ\cdots\circ\psi_{i_h}(x_0) & i_1,\dots,i_h \in \{1,2,3,4\}, \\ B_{i_1,\dots,i_h} = x_{i_1,\dots,i_h} + r_h B(0,1), \\ T_{i_1,\dots,i_h} = B_{i_1,\dots,i_h} \cap \Sigma, \\ T_h = \bigcup_{i_1,\dots,i_h\in\{1,2,3,4\}} T_{i_1,\dots,i_h}, \end{cases}$$

where $r_h = \exp\left(-3^{hd}/c\right)$ for a given positive constant c. We define the space

$$W_{c,h} = \left\{u \in H^1\left(\Omega\backslash\Sigma;\mathbb{R}^2\right) \mid [u]_\Sigma = 0 \text{ on } T_h \text{ and } u = 0 \text{ on } \partial\Omega\right\},$$

where $[u]_\Sigma$ is the jump of u across Σ. The trace of $u \in H^1(\Omega\backslash\Sigma)$ on Σ exists for $\mathcal{H}^d$-a.e. $x \in \Sigma$ and belongs to the Besov space $B_d(\Sigma)$ defined as

$$B_d(\Sigma) = \left\{v : \Sigma \to \mathbb{R}^2 \mid \int_\Sigma |u|^2 d\mathcal{H}^d + \int_{\substack{\Sigma\times\Sigma \\ |x-y|<1}} \frac{|u(x)-u(y)|^2}{|x-y|^{2d}} d\mathcal{H}^d < \infty\right\},$$

see [Wa91] for more details. Given $f \in L^2\left(\Omega;\mathbb{R}^2\right)$, we consider the following problem:

$$\min_{u\in L^2(\Omega;\mathbb{R}^n)}\left\{F_{c,h}(u)-2\int_\Omega f\cdot udx\right\},\tag{8.8}$$

where the functional $F_{c,h}$ is defined on $L^2\left(\Omega;\mathbb{R}^2\right)$ through

$$F_{c,h}(u)=\begin{cases}\int_{\Omega\setminus\Sigma}\sigma_{ij}(u)\,e_{ij}(u)\,dx & \text{if } u\in W_{c,h},\\ +\infty & \text{otherwise.}\end{cases}$$

The problem (8.8) has a unique solution $u^h\in W_{c,h}$ with $\left\|u^h\right\|_{H^1(\Omega\setminus\Sigma;\mathbb{R}^2)}\leq C$, where C is some positive constant independent of h. Let $(y_1,y_{2,\Sigma})$ be an element of Σ. We define $\mathbb{R}^2_\Sigma$ as $\mathbb{R}^2_\Sigma=\{(y_1,y_2)\in\mathbb{R}^2\mid y_2>y_{2,\Sigma}$ or $y_2<y_{2,\Sigma}$, $\forall(y_1,y_{2,\Sigma})\in\Sigma\}$.

We consider the following problem:

$$\begin{cases}-\sigma_{ij,j}\left(w^m_\Sigma\right) &= 0 & \text{in } \mathbb{R}^2_\Sigma,\ i,m=1,2,\\ w^m_\Sigma &= e_m & \text{on } \Sigma,\\ \left(w^m_\Sigma\right)_i &= \delta_{im}\ln(|y|)+O(1) & \text{as } |y|\to\infty,\\ \left|\left(w^m_\Sigma\right)_p\right| &\leq C & \text{for } p\neq m.\end{cases}\tag{8.9}$$

Here, w^m_Σ can be computed using the complex potentials of Kolosov–Muskhelishvili (see [Mu63]) and the conformal mapping from $\mathbb{R}^2_\Sigma$ to $\mathbb{R}^2\setminus[0,1]$ (see [ElBr08] for more details). The elastic energy associated to (8.9) verifies

$$\begin{aligned}&\lim_{R\to\infty}\tfrac{1}{\ln(R)}\int_{\mathbb{R}^2_\Sigma\cap B(0,R)}\sigma_{ij}\left(w^m_\Sigma\right)e_{ij}\left(w^l_\Sigma\right)dx\\ &=\delta_{lm}\tfrac{4\mu}{(\chi+1)}\int_\Sigma q(s)\,d\mathcal{H}^d(s),\end{aligned}\tag{8.10}$$

where $\frac{4\mu}{(\chi+1)}q(s)\,d\mathcal{H}^d(s)=\sigma_{mj}\left(w^m_\Sigma\right)\nu_j\mid_\Sigma$, ν is the unit normal to Σ directed outward the region $\{y_2>y_{2,\Sigma}\}$. The normal strains $\sigma_{mj}\left(w^m_\Sigma\right)\nu_j\mid_\Sigma$ belong to the dual space of $B_d(\Sigma)$ (see [ElBr08], [JoWa95], for example). We define the functions $w^{h,m}_\Sigma$ as $w^{h,m}_\Sigma(x)=-\frac{1}{\ln(r_h)}\left(w^m_\Sigma\left((x-x_{i_1\cdots i_h})/r_h\right)-e_m\right)$ and the space $W_{c,\infty}=\left\{u\in H^1\left(\Omega\setminus\Sigma;\mathbb{R}^2\right)\mid[u]_\Sigma\in B_d\left(\Sigma;\mathbb{R}^2\right),\ u=0\text{ on }\partial\Omega\right\}$.

Our main result in this section is the following.

Theorem 2. *The sequence* $\left(F_{c,h}\right)_h$ Γ*-converges to the functional* $F_{c,\infty}$ *defined through*

$$F_{c,\infty}(u)=\begin{cases}\int_{\Omega\setminus\Sigma}\sigma_{ij}(u)\,e_{ij}(u)\,dx\\ \quad+c\frac{\mu\gamma_d}{(\chi+1)\mathcal{H}^d(\Sigma)}\int_\Sigma|[u]|^2\,d\mathcal{H}^d & \textit{if } u\in W_{c,\infty},\\ +\infty & \textit{otherwise,}\end{cases}$$

with respect to the strong topology of $L^2\left(\Omega;\mathbb{R}^2\right)$*. Here* $\gamma_d=\int_\Sigma q(s)\,d\mathcal{H}^d(s)$.

Proof. Let $s_h=(1/3)^{hd}$. We choose a smooth truncation function $\varphi_{i_1,\cdots,i_h}$ for $i_1,\ldots,i_h\in\{1,2,3,4\}$ satisfying

$$\varphi_{i_1,\ldots,i_h}(x) = \begin{cases} 1 & \text{in } B\left(x_{i_1,\ldots,i_h}, \frac{s_h}{2}\right), \\ 0 & \text{in } \Omega \backslash B\left(x_{i_1,\ldots,i_h}, s_h\right). \end{cases}$$

There exist two open subsets Ω^1 and Ω^2 such that $\Omega \backslash \Sigma = \Omega^1 \cup \Lambda \cup \Omega^2$, with $\Lambda = \mathbb{R} \backslash [0,1]$. Let $u \in C^1\left(\overline{\Omega} \backslash \Sigma; \mathbb{R}^2\right)$ be such that $u = 0$ on $\partial\Omega$. We choose one of the regular images of $\frac{1}{2}[u]_\Sigma$ (resp. $-\frac{1}{2}[u]_\Sigma$) through the continuous mapping r_Σ^1 from $B_d\left(\Sigma; \mathbb{R}^2\right)$ into $H^1\left(\Omega^1; \mathbb{R}^2\right)$ (resp. r_Σ^2 from $B_d\left(\Sigma; \mathbb{R}^2\right)$ into $H^1\left(\Omega^2; \mathbb{R}^2\right)$), respectively denoted by $r_\Sigma^1\left(\frac{1}{2}[u]_\Sigma\right)$ and $r_\Sigma^2\left(-\frac{1}{2}[u]_\Sigma\right)$. We define the function u_0^h as follows, for every $i_1, \ldots, i_h \in \{1,2,3,4\}$:

$$u_0^h = \begin{cases} u\left(1 - \varphi_{i_1,\ldots,i_h}\right) & \\ +\varphi_{i_1,\ldots,i_h} w_\Sigma^{h,m} r_\Sigma^1\left(\frac{1}{2}[u_m]_\Sigma\right) & \text{in } B\left(x_{i_1,\ldots,i_h}, s_h\right) \cap \Omega^1, \\ u\left(1 - \varphi_{i_1,\ldots,i_h}\right) & \\ +\varphi_{i_1,\ldots,i_h} w_\Sigma^{h,m} r_\Sigma^2\left(-\frac{1}{2}[u_m]_\Sigma\right) & \text{in } B\left(x_{i_1,\ldots,i_h}, s_h\right) \cap \Omega^2, \\ u & \text{in } \Omega \backslash \underset{i_1,\ldots,i_h \in \{1,2,3,4\}}{\cup} B\left(x_{i_1,\ldots,i_h}, s_h\right). \end{cases}$$

It is easily seen that $u_0^h \in W_{c,h}$ and $(u_0^h)_h$ converges to u in $L^2\left(\Omega; \mathbb{R}^2\right)$-strong. On the other hand, one has

$$\begin{aligned} F_{c,h}\left(u_0^h\right) &= \int_{\Omega \backslash \Sigma} \sigma_{ij}\left(u_0^h\right) e_{ij}\left(u_0^h\right) dx = \int_{\Omega \backslash \Sigma} \sigma_{ij}(u)\, e_{ij}(u)\, dx \\ &\quad + \sum_{i_1,\ldots,i_h \in \{1,2,3,4\}} \frac{-1}{\ln(r_h)} \frac{[u_m][u_l]\left(x_{i_1,\cdots,i_h}\right)}{4} \\ &\quad \times \left(\frac{-1}{\ln(r_h)} \int_{\left(B_h(s_h/r_h) \cap \mathbb{R}^2_\Sigma\right)} \sigma_{ij}\left(w_\Sigma^m\right) e_{ij}\left(w_\Sigma^l\right) dy\right) + o\left(\frac{1}{h}\right). \end{aligned}$$

Using (8.10), we get

$$\begin{aligned} \lim_{h \to \infty} F_{c,h}\left(u_0^h\right) &= \int_{\Omega \backslash \Sigma} \sigma_{ij}(u)\, e_{ij}(u)\, dx + c \frac{\mu \gamma_d}{(\chi+1)\mathcal{H}^d(\Sigma)} \int_\Sigma |[u]|^2\, d\mathcal{H}^d \\ &= F_{c,\infty}(u). \end{aligned}$$

Using the same method as in the proof of Theorem 1, we conclude that, for every $u \in L^2\left(\Omega; \mathbb{R}^2\right)$ $\Gamma\text{-}\lim_{h\to\infty} F_{c,h}(u) = F_{c,\infty}(u)$. We then end the proof in a similar way as in the proof of Theorem 1.

Remark 1. Another interfacial problem may deal with a contact situation in granular materials. The non-overlapping spherical elastic grains are supposed to be confined in some bounded domain Ω, and perfect adhesion between seeds occurs on thin zones disposed along a self-similar fractal K. The asymptotic relaxed energy is proved to involve an integral extra term of the form $\int_{\Sigma \cap K} A(x)\, [u]_\Sigma \cdot [u]_\Sigma\, d\mathcal{H}^d(x)$, where Σ is the union of the boundaries of the grains and $A(x)$ is a symmetric matrix depending on the position x and on the material coefficients of the problem. We can also consider a contact problem on spheres in the Apollonian packing. Here Σ is a fractal set which is not self-similar and whose fractal dimension d has been numerically determined in [BoDePe94]. For every $h \in \mathbb{N}^*$, we suppose that a perfect adhesion occurs on thin zones between the $N(h)$ balls of radii larger than the radius ρ_h of some fixed ball B_h. Using the asymptotic relation $N(h) \sim \rho_h^{-d}$ (see [Bo73]), we can prove that the relaxed energy takes the above form with an integral covering the whole Σ.

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9

Hyers–Ulam and Hyers–Ulam–Rassias Stability of Volterra Integral Equations with Delay

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9.1 Introduction

Considerable attention has been given to the study of the Hyers–Ulam and Hyers–Ulam–Rassias stability of functional equations (see, e.g., [HIR98, Ju01]). The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus, the stability question of functional equations is *how do the solutions of the inequality differ from those of the given functional equation?*

Although there are numerous publications for different types of equations, there are very few results on the study of these kinds of stabilities for integral equations (cf. [CaRa09] and [Ju07]). In this chapter we propose both a Hyers–Ulam and a Hyers–Ulam–Rassias stability study for the delay Volterra-type integral equations [Bu83, Co88, GLS90, LaRa95] of the form

$$y(x) = \int_c^x f(x,\tau,y(\tau),y(\alpha(\tau)))\,d\tau \qquad (-\infty < a \le x \le b < +\infty), \tag{9.1}$$

where a, b, and c are fixed real numbers such that $a < b$ and $c \in (a,b)$, $f : [a,b] \times [a,b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function, and $\alpha : [a,b] \to [a,b]$ is a continuous delay function which therefore fulfills $\alpha(x) \le x$, for all $x \in [a,b]$.

We would like to recall that the kinds of stability which we are studying here (for the above integral equation) appeared for the first time in 1941 when Hyers [Hy41] proved the following result by answering a problem of Ulam affirmatively; cf. [Ul60] and [Ul74]): *Let S_1 and S_2 be two (real) Banach spaces and assume that a mapping $h : S_1 \to S_2$ satisfies the inequality*

$$\|h(x+y) - h(x) - h(y)\| \le \epsilon \qquad (x, y \in S_1) \tag{9.2}$$

for some nonnegative ϵ. Then there is a (unique) additive mapping $\mathcal{A} : S_1 \to S_2$ such that

$$\|\mathcal{A}(x) - h(x)\| \le \epsilon \qquad (x \in S_1)$$

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holds. In addition, it was also proved in [Hy41] that $\mathcal{A}(x) = \lim_{n\to\infty} h(2^n x)/2^n$ $(x \in S_1)$.

The last result is nowadays called the *Hyers–Ulam stability theorem* (of the additive Cauchy equation $f(x+y) = f(x) + f(y)$). Since Hyers's result, numerous papers on the subject have been published, extending and generalizing Ulam's problem and Hyers's theorem in various directions. One of these new directions was introduced by Th. M. Rassias [Ra7] by considering unbounded right-hand sides in (9.2) which depend on certain functions of x and y (instead of considering only bounded Cauchy differences $f(x+y) - f(x) - f(y)$ as in the Hyers case).

In this chapter, the formal definitions of the above-mentioned two types of stability for the case of the equation (9.1) can be defined as follows. If for each function y satisfying

$$\left| y(x) - \int_c^x f(x, \tau, y(\tau), y(\alpha(\tau)))\, d\tau \right| \leq \sigma(x)$$

(where σ is a nonnegative function), there is a solution y_0 of the Volterra integral equation (9.1) and a constant $C_1 > 0$ independent of y and y_0 such that

$$|y(x) - y_0(x)| \leq C_1\, \sigma(x),$$

for all x, then we say that the integral equation (9.1) has the *Hyers–Ulam–Rassias stability.* In the case where σ takes the form of a constant function, we say that the integral equation (9.1) has the *Hyers–Ulam stability.*

The interested reader can find further details about Hyers–Ulam stability of functional equations in the extensive survey [Fo95].

9.2 The Hyers–Ulam–Rassias Stability of the Volterra Integral Equation with Delay

This section is devoted to studying conditions under which the Volterra integral equation with delay (9.1) admits the Hyers–Ulam–Rassias stability.

Banach's fixed point theorem will be one of the main ideas upon which such properties will be obtained. Here, we will use this theorem in a framework of a generalized complete metric space setting (Y, d_Y). We recall that a function $d_Y : Y \times Y \to [0, +\infty]$ is called a *generalized metric on* Y if and only if d_Y satisfies the following three properties:

(i) $d_Y(x,y) = 0$ if and only if $x = y$;

(ii) $d_Y(x,y) = d_Y(y,x)$ for all $x, y \in Y$;

(iii) $d_Y(x,z) \leq d_Y(x,y) + d_Y(y,z)$ for all $x, y, z \in Y$.

Having a generalized complete metric space (Y, d_Y), we will denote by $\mathrm{Con}(Y)$ the set of (strict) contraction operators on the space Y, i.e.,

$$\begin{aligned}\mathrm{Con}(Y) \quad := \quad & \{T : Y \to Y \mid d_Y(Ty_1, Ty_2) \le c_T\, d_Y(y_1, y_2), \\ & \qquad \text{for all } y_1, y_2 \in Y \text{ and for some } c_T \in [0,1)\}.\end{aligned}$$

Theorem 1 (Banach). *Let (Y, d_Y) be a generalized complete metric space and consider $T \in Con(Y)$ having a Lipschitz constant $c_T < 1$. If there is a nonnegative integer k such that $d(T^{k+1}y, T^k y) < \infty$ for some $y \in Y$, then the following propositions hold true:*

(i) the sequence $(T^n y)_{n\in\mathbb{N}}$ converges to a fixed point y^* of T;
(ii) y^* is the unique fixed point of T in

$$Y^* = \{z \in Y \mid d(T^k y, z) < \infty\};$$

(iii) if $z \in Y^*$, then

$$d(z, y^*) \le \frac{1}{1 - c_T}\, d(Tz, z).$$

Proposition (iii) in the last result is referred to as the *collage theorem* in the fractals literature.

9.2.1 The Compact Interval Case

We now have the instruments to present sufficient conditions for the Hyers–Ulam–Rassias stability of the Volterra integral equation with delay (9.1), where $x \in [a, b]$ for some fixed real numbers a and b.

Theorem 2. *Let C and L be positive constants with $0 < CL < 1$ and assume that $\alpha : [a, b] \to [a, b]$ is a continuous function such that*

$$\alpha(x) \le x, \quad \textit{for all} \quad x \in [a, b]$$

and $f : [a, b] \times [a, b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function which additionally satisfies the Lipschitz condition

$$|f(x, \tau, y_1(\tau), y_1(\alpha(\tau))) - f(x, \tau, y_2(\tau), y_2(\alpha(\tau)))| \le L|y_1 - y_2| \tag{9.3}$$

for any $x, \tau \in [a, b]$ and all $y_1, y_2 \in \mathbb{C}$.

If a continuous function $y : [a, b] \to \mathbb{C}$ satisfies

$$\left| y(x) - \int_c^x f(x, \tau, y(\tau), y(\alpha(\tau)))\, d\tau \right| \le \varphi(x) \tag{9.4}$$

for all $x \in [a, b]$ and for some $c \in (a, b)$, where $\varphi : [a, b] \to (0, \infty)$ is a continuous function with

$$\left| \int_c^x \varphi(\tau)\, d\tau \right| \le C\varphi(x) \tag{9.5}$$

for each $x \in [a,b]$, *then there is a unique continuous function* $y_0 : [a,b] \to \mathbb{C}$ *such that*

$$y_0(x) = \int_c^x f(x, \tau, y(\tau), y(\alpha(\tau)))\, d\tau \tag{9.6}$$

$$|y(x) - y_0(x)| \leq \frac{1}{1 - CL} \varphi(x) \tag{9.7}$$

for all $x \in [a,b]$.

Proof. We will consider the space of continuous functions

$$X = \{g : [a,b] \to \mathbb{C} \mid g \text{ is continuous}\} \tag{9.8}$$

endowed with the generalized metric defined by

$$d(g,h) = \inf\{C \in [0,\infty] \mid |g(x) - h(x)| \leq C\varphi(x)\,,\ \text{for all } x \in [a,b]\}.$$

It is known that (X,d) is a complete generalized metric space (cf., e.g., [Ju07]).

We will consider the following operator $T : X \to X$, defined by

$$(Tg)(x) = \int_c^x f(x, \tau, y(\tau), y(\alpha(\tau)))\, d\tau$$

for all $g \in X$ and $x \in [a,b]$. Thus, due to the fact that f is a continuous function, it follows that Tg is also continuous and this ensures that T is a well-defined operator. Indeed,

$$\begin{aligned}
&|(Tg)(x) - (Tg)(x_0)| \\
&\quad = \left| \int_c^x f(x,\tau,g(\tau),g(\alpha(\tau)))\,d\tau - \int_c^{x_0} f(x_0,\tau,g(\tau),g(\alpha(\tau)))\,d\tau \right| \\
&\quad = \left| \int_c^x f(x,\tau,g(\tau),g(\alpha(\tau)))\,d\tau - \int_c^{x} f(x_0,\tau,g(\tau),g(\alpha(\tau)))\,d\tau \right. \\
&\qquad\quad \left. + \int_c^x f(x_0,\tau,g(\tau),g(\alpha(\tau)))\,d\tau - \int_c^{x_0} f(x_0,\tau,g(\tau),g(\alpha(\tau)))\,d\tau \right| \\
&\quad \leq \left| \int_c^x f(x,\tau,g(\tau),g(\alpha(\tau)))\,d\tau - \int_c^{x} f(x_0,\tau,g(\tau),g(\alpha(\tau)))\,d\tau \right| \\
&\qquad\quad + \left| \int_c^x f(x_0,\tau,g(\tau),g(\alpha(\tau)))\,d\tau - \int_c^{x_0} f(x_0,\tau,g(\tau),g(\alpha(\tau)))\,d\tau \right| \\
&\quad \leq \int_c^x |f(x,\tau,g(\tau),g(\alpha(\tau))) - f(x_0,\tau,g(\tau),g(\alpha(\tau)))|\,d\tau \\
&\qquad\quad + \left| \int_{x_0}^x f(x_0,\tau,g(\tau),g(\alpha(\tau)))\,d\tau \right| \quad \overset{x \to x_0}{\longrightarrow} \quad 0.
\end{aligned}$$

The main reason to introduce the operator T is to make the application of Theorem 1 possible, and so let us now verify that T is strictly contractive on X. For any $g, h \in X$, let us consider $C_{gh} \in [0,\infty]$ such that

$$|g(x) - h(x)| \leq C_{gh}\varphi(x) \tag{9.9}$$

for any $x \in [a, b]$ (note that this is always possible due to the definition of (X, d)). From the definition of T and (9.3), (9.5), and (9.9), it follows that

$$\begin{aligned}
|(Tg)(x) - (Th)(x)| &= \left| \int_c^x [f(x, \tau, g(\tau), g(\alpha(\tau))) - f(x, \tau, h(\tau), h(\alpha(\tau)))]\, d\tau \right| \\
&\leq \left| \int_c^x |f(x, \tau, g(\tau), g(\alpha(\tau))) - f(x, \tau, h(\tau), h(\alpha(\tau)))|\, d\tau \right| \\
&\leq L \left| \int_c^x |g(\tau) - h(\tau)|\, d\tau \right| \\
&\leq LC_{gh} \left| \int_c^x \varphi(\tau)\, d\tau \right| \\
&\leq LC_{gh} C\, \varphi(x)
\end{aligned}$$

for all $x \in [a, b]$. Therefore,

$$d(Tg, Th) \leq LC_{gh}C.$$

This allows us to conclude that $d(Tg, Th) \leq LCd(g, h)$ for any $g, h \in X$, and since $CL \in (0, 1)$ the (strictly) contraction property is verified.

Let us take $g_0 \in X$. From the continuous property of g_0 and Tg_0, it follows that there is a constant $C_1 \in (0, \infty)$ such that

$$\begin{aligned}
|(Tg_0)(x) - g_0(x)| &= \left| \int_c^x f(x, \tau, g_0(\tau), g_0(\alpha(\tau))\,)d\tau - g_0(x) \right| \\
&\leq C_1\varphi(x)
\end{aligned}$$

for all $x \in [a, b]$. Note that this occurs also because f and g_0 are bounded on $[a, b]$ and φ is a positive function. Therefore, from the definition of the generalized metric d, it follows that

$$d(Tg_0, g_0) < \infty. \tag{9.10}$$

In this way, we are ready to use Theorem 1 and so to conclude that there is a continuous function $y_0 : [a, b] \to \mathbb{C}$ such that

$$T^n g_0 \overset{n\to\infty}{\longrightarrow} y_0 \quad \text{in} \quad (X, d),$$

and $Ty_0 = y_0$.

For any g_0 with the property (9.10) it follows that X can be rewritten in the following new form:

$$X = \{g \in X \,|\, d(g_0, g) < \infty\}$$

(cf. [Ju07]). Therefore, once again Theorem 1 ensures that y_0 is the unique continuous function with the property (9.6).

Now, from (9.4) it follows that $d(y, Ty) \le 1$, and so the collage theorem leads to

$$d(y, y_0) \le \frac{1}{1 - CL} d(Ty, y) \le \frac{1}{1 - CL}.$$

Thus, the last inequality together with the definition of the generalized metric d lead to inequality (9.7).

9.2.2 The Infinite Interval Case

In this subsection we will consider a modification of the Volterra integral equation with delay (9.1) to the situation of infinite intervals instead of the compact case presented in the Introduction. Here the case $x \in \mathbb{R}$ (instead of the above case of $x \in [a, b]$ with fixed real numbers a and b) will be dealt with in detail. The corresponding cases of $x \in [a, +\infty)$ and $x \in (-\infty, b]$ also hold true by applying obvious changes in the strategy below.

The main goal here is also to obtain the Hyers–Ulam–Rassias stability of such (different) corresponding integral equations. In view of this, our strategy will be based on the application of a recurrence procedure due to the already-obtained result for the above-studied compact interval case.

Theorem 3. *Let C and L be positive constants with $0 < CL < 1$ and assume that*

$$f : \mathbb{R} \times \mathbb{R} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$$

is a continuous function which additionally satisfies the Lipschitz condition (9.3), for any $x, \tau \in \mathbb{R}$ and all $y_1, y_2 \in \mathbb{C}$, and $\alpha : \mathbb{R} \to \mathbb{R}$ is also a continuous function such that

$$\alpha(x) \le x, \quad \text{for all} \quad x \in \mathbb{R}.$$

If a continuous function $y : \mathbb{R} \to \mathbb{C}$ satisfies (9.4), for all $x \in \mathbb{R}$ and for some $c \in \mathbb{R}$, where $\varphi : \mathbb{R} \to (0, \infty)$ is a continuous function satisfying (9.5), for each $x \in \mathbb{R}$, then there is a unique continuous function

$$y_0 : \mathbb{R} \to \mathbb{C}$$

which satisfies (9.6) and (9.7) for all $x \in \mathbb{R}$.

Proof. We start by proving that y_0 is a continuous function. For any $n \in \mathbb{N}$, let us define $I_n = [c - n, c + n]$. According to Theorem 2, there is a unique continuous function $y_{0,n} : I_n \to \mathbb{C}$ such that

$$y_{0,n}(x) = \int_c^x f(x, \tau, y_{0,n}(\tau), y_{0,n}(\alpha(\tau)))\, d\tau \tag{9.11}$$

$$|y(x) - y_{0,n}(x)| \le \frac{1}{1 - CL} \varphi(x) \tag{9.12}$$

for all $x \in I_n$, where α is defined on I_n. The uniqueness of $y_{0,n}$ implies that if $x \in I_n$ then

$$y_{0,n}(x) = y_{0,n+1}(x) = y_{0,n+2}(x) = \cdots . \tag{9.13}$$

For any $x \in \mathbb{R}$, let us define $n(x) \in \mathbb{N}$ as

$$n(x) = \min\{n \in \mathbb{N} \mid x \in I_n\}.$$

We also define a function $y_0 : \mathbb{R} \to \mathbb{C}$ by

$$y_0(x) = y_{0,n(x)}(x),$$

and we can say that y_0 is continuous. Indeed, for any $x_1 \in \mathbb{R}$, let $n_1 = n(x_1)$. Thus, x_1 belongs to the interior of I_{n_1+1} and an $\epsilon > 0$ exists such that $y_0(x) = y_{0,n_1+1}(x)$ for all $x \in (x_1 - \epsilon, x_1 + \epsilon)$. By Theorem 2, y_{0,n_1+1} is continuous at x_1, so it is y_0.

In the next step, we will show that y_0 satisfies (9.6) and (9.7) for all $x \in \mathbb{R}$. Let us choose $n(x)$ for an arbitrary $x \in \mathbb{R}$. Then $x \in I_{n(x)}$ and from (9.11) it follows that

$$\begin{aligned} y_0(x) &= y_{0,n(x)}(x) \\ &= \int_c^x f(x, \tau, y_{0,n(x)}(\tau), y_{0,n(x)}(\alpha(\tau)))\, d\tau \\ &= \int_c^x f(x, \tau, y_0(\tau), y_0(\alpha(\tau)))\, d\tau \end{aligned}$$

(where the last equality holds true because $n(\tau) \le n(x)$ and $n(\alpha(\tau)) \le n(x)$, for any $\tau \in I_{n(x)}$), and it follows from (9.13) that

$$y_0(\tau) = y_{0,n(\tau)}(\tau) = y_{0,n(x)}(\tau)$$

and

$$y_0(\alpha(\tau)) = y_{0,n(\tau)}(\alpha(\tau)) = y_{0,n(x)}(\alpha(\tau)).$$

Moreover, (9.12) implies that

$$|y(x) - y_0(x)| = \left|y(x) - y_{0,n(x)}(x)\right| \le \frac{1}{1-CL}\varphi(x), \quad \text{for all} \quad x \in \mathbb{R}.$$

We will now prove that y_0 is unique. Suppose that y_1 is another continuous function which satisfies (9.6) and (9.7), for all $x \in \mathbb{R}$. Since both restrictions $y_{0|I_{n(x)}} = y_{0,n(x)}$ and $y_{1|I_{n(x)}}$ satisfy (9.6) and (9.7) for all $x \in I_{n(x)}$, the uniqueness of $y_{0|I_{n(x)}} = y_{0,n(x)}$ implies that

$$y_0(x) = y_{0|I_{n(x)}(x)} = y_{1|I_{n(x)}(x)} = y_1(x).$$

9.3 The Hyers–Ulam Stability of the Volterra Integral Equation with Delay

We would now like to consider certain stronger assumptions in the conditions associated with the Volterra integral equation with delay (9.1) (for the finite interval case) such that somehow the Hyers–Ulam stability will be obtained. All this is gathered in the next final result.

Theorem 4. *Let $K = b - a$ and consider L to be a positive constant such that $0 < KL < 1$. Assume in addition that*

$$\alpha : [a, b] \to [a, b]$$

is a continuous function such that $\alpha(x) \leq x$, for all $x \in [a, b]$, and

$$f : [a, b] \times [a, b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$$

is a continuous function which fulfills the Lipschitz condition

$$|f(x, \tau, y_1(\tau), y_1(\alpha(\tau))) - f(x, \tau, y_2(\tau), y_2(\alpha(\tau)))| \leq L|y_1 - y_2| \tag{9.14}$$

for any $x, \tau \in [a, b]$ and all $y_1, y_2 \in \mathbb{C}$.

If for some $c \in (a, b)$ a continuous function $y : [a, b] \to \mathbb{C}$ satisfies

$$\left| y(x) - \int_c^x f(x, \tau, y(\tau), y(\alpha(\tau)))\, d\tau \right| \leq \theta$$

for each $x \in [a, b]$ and some $\theta \geq 0$, then a unique continuous function $y_0 : [a, b] \to \mathbb{C}$ exists such that

$$y_0(x) = \int_c^x f(x, \tau, y_0(\tau), y_0(\alpha(\tau)))\, d\tau \tag{9.15}$$

and

$$|y(x) - y_0(x)| \leq \frac{\theta}{1 - KL}$$

for all $x \in [a, b]$.

Proof. We will continue working with the space of continuous functions presented in (9.8) and endowed with the generalized metric defined by

$$d(g, h) = \inf\{C \in [0, \infty] \mid |g(x) - h(x)| \leq C\,, \text{ for all } x \in [a, b]\},$$

and consider also the operator $T : X \to X$ defined by

$$(Tg)(x) = \int_c^x f(x, \tau, g(\tau), g(\alpha(\tau)))\, d\tau$$

for all $g \in X$ and $x \in [a,b]$. We recall that for any continuous function g, the element Tg is also continuous.

Let us now verify that $T \in \mathrm{Con}(X)$. For any $g, h \in X$, let us consider $C_{gh} \in [0, \infty]$ such that

$$|g(x) - h(x)| \leq C_{gh} \tag{9.16}$$

for any $x \in [a,b]$. From the definition of T, (9.14), and (9.16), it follows that

$$\begin{aligned}
|(Tg)(x) - (Th)(x)| &= \left| \int_c^x [f(x,\tau,g(\tau),g(\alpha(\tau))) - f(x,\tau,h(\tau),h(\alpha(\tau)))]\, d\tau \right| \\
&\leq \left| \int_c^x |f(x,\tau,g(\tau),g(\alpha(\tau))) - f(x,\tau,h(\tau),h(\alpha(\tau)))|\, d\tau \right| \\
&\leq L \left| \int_c^x |g(\tau) - h(\tau)|\, d\tau \right| \\
&\leq L C_{gh} K
\end{aligned}$$

for all $x \in [a,b]$. Thus, $d(Tg,Th) \leq LC_{gh}K$. This allows us to conclude that $d(Tg,Th) \leq LKd(g,h)$ for any $g,h \in X$, and since $KL \in (0,1)$ the (strict) contraction property is verified.

In an analogous way to the proof of Theorem 2, we can choose $g_0 \in X$ with

$$d(Tg_0, g_0) < \infty. \tag{9.17}$$

Therefore, we are in the condition of using Theorem 1 and thus conclude that there is a continuous function $y_0 : [a,b] \to \mathbb{C}$ such that

$$T^n g_0 \overset{n\to\infty}{\longrightarrow} y_0 \quad \text{in} \quad (X,d),$$

and $Ty_0 = y_0$.

For any g_0 with the property (9.17) it follows that X can be rewritten in the following new form:

$$X = \{g \in X \,|\, d(g_0, g) < \infty\}.$$

Thus, once again Theorem 1 ensures that y_0 is the unique continuous function with the property (9.15). Furthermore, the *collage theorem* (cf. Theorem 1) yields

$$|y(x) - y_0(x)| \leq \frac{\theta}{1 - KL},$$

for all $x \in [a,b]$.

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10

Fredholm Index Formula for a Class of Matrix Wiener–Hopf Plus and Minus Hankel Operators with Symmetry

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10.1 Introduction

The main goal of this chapter is to obtain a Fredholm index formula for a class of Wiener–Hopf plus and minus Hankel operators which contain a certain symmetry between their Fourier symbols. It is relevant to mention that Wiener–Hopf plus and minus Hankel operators (with and without symmetries) appear in several different kinds of applications [CST04]; therefore, further knowledge about their Fredholm property and index is relevant for both theoretical and applied reasons. In view of this, several works concerning these classes of operators have appeared recently [BoCa06, BoCa, CaSi09, NoCa07]. The Fourier matrix symbols considered in this chapter belong to the C^*-algebra of piecewise almost periodic functions. Besides the Fredholm index formula, conditions that ensure the Fredholm property of the operators under study will also be obtained.

Let us now define in exact terms the operators which we will be working with. We will be concerned with matrix integral operators which have the following diagonal form:

$$\mathfrak{D}_\Upsilon = \mathrm{diag}\Big[W_{\widetilde{\Upsilon}} + H_\Upsilon, W_{\widetilde{\Upsilon}} - H_\Upsilon\Big] : [L^2_+(\mathbb{R})]^{2N} \to [L^2(\mathbb{R}_+)]^{2N}, \qquad (10.1)$$

where in the main diagonal we find matrix Wiener–Hopf plus and minus Hankel operators

$$W_{\widetilde{\Upsilon}} \pm H_\Upsilon \;:\; [L^2_+(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^N \qquad (N \in \mathbb{N}), \qquad (10.2)$$

where $W_{\widetilde{\Upsilon}}$ and H_Υ are matrix Wiener–Hopf and Hankel operators defined by $W_{\widetilde{\Upsilon}} = r_+\mathcal{F}^{-1}\widetilde{\Upsilon} \cdot \mathcal{F}$ and $H_\Upsilon = r_+\mathcal{F}^{-1}\Upsilon \cdot \mathcal{F}J$, respectively. In addition, $\mathcal{F}$ denotes the Fourier transformation, $\widetilde{\varphi}(x) = \varphi(-x)$, $x \in \mathbb{R}$, and J is the reflection operator given by the rule $J\varphi = \widetilde{\varphi}$. We use $[L^2_+(\mathbb{R})]^N$ to denote the subspace of $[L^2(\mathbb{R})]^N$ formed by all the matrix functions supported on the closure of $\mathbb{R}_+ = (0, +\infty)$, r_+ represents the operator of restriction from

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$[L^2_+(\mathbb{R})]^N$ into $[L^2(\mathbb{R}_+)]^N$, and $\widetilde{\Upsilon}$, Υ are called the Fourier $N \times N$ matrix symbols (which will belong to the above-mentioned C^*-algebra of piecewise almost periodic elements).

10.2 Auxiliary Material

In view of defining the piecewise almost periodic functions, we will first consider the algebra of almost periodic functions.

The smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions e_λ ($\lambda \in \mathbb{R}$), where $e_\lambda(x) = e^{i\lambda x}$, $x \in \mathbb{R}$, is denoted by AP and called the algebra of *almost periodic functions*: $AP := alg_{L^\infty(\mathbb{R})}\{e_\lambda : \lambda \in \mathbb{R}\}$. In addition, we will also use $AP_+ := alg_{L^\infty(\mathbb{R})}\{e_\lambda : \lambda \geq 0\}$, and $AP_- := alg_{L^\infty(\mathbb{R})}\{e_\lambda : \lambda \leq 0\}$.

We will review here some of the properties of the almost periodic functions (which we will use further on). Let $A \subset (0, \infty)$ be an unbounded set and let $\{I_\alpha\}_{\alpha \in A} = \{(x_\alpha, y_\alpha)\}_{\alpha \in A}$ be a family of intervals $I_\alpha \subset \mathbb{R}$ such that $|I_\alpha| = y_\alpha - x_\alpha \to \infty$ as $\alpha \to \infty$. If $\varphi \in AP$, then the limit $M(\varphi) := \lim_{\alpha\to\infty} \frac{1}{|I_\alpha|} \int_{I_\alpha} \varphi(x)\, dx$ exists, is finite, and is independent of the particular choice of the family $\{I_\alpha\}$ (cf., e.g., [BKS02], Proposition 2.22). The number $M(\varphi)$ is called the *Bohr mean value* or simply the mean value of φ. In the matrix case the *mean value* is defined entry-wise.

Let $C(\dot{\mathbb{R}})$ (with $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$) denote the set of all (bounded and) continuous functions φ on the real line for which the two limits $\varphi(-\infty) := \lim_{x\to-\infty} \varphi(x)$, $\varphi(+\infty) := \lim_{x\to+\infty} \varphi(x)$ exist and coincide. The common value of these two limits will be denoted by $\varphi(\infty)$. In addition, consider the C^*-algebra of all bounded piecewise continuous functions on $\dot{\mathbb{R}}$ denoted by PC or $PC(\dot{\mathbb{R}})$ as being the algebra of all functions $\varphi \in L^\infty(\mathbb{R})$ for which the one-sided limits $\varphi(x_0 - 0) = \lim_{x\to x_0-0} \varphi(x)$, $\varphi(x_0 + 0) = \lim_{x\to x_0+0} \varphi(x)$ exist for each $x_0 \in \dot{\mathbb{R}}$. $C(\overline{\mathbb{R}}) := C(\mathbb{R}) \cup PC(\dot{\mathbb{R}})$, where $C(\mathbb{R})$ is the usual set of continuous functions on the real line. Furthermore, PC_0 will represent the subclass of PC of all piecewise continuous functions φ for which $\varphi(\pm\infty) = 0$.

As mentioned above, we will deal with Fourier symbols from the C^*-algebra of piecewise almost periodic elements which is defined as follows.

Definition 1. *The C^*-algebra PAP of all* piecewise almost periodic functions *on $\mathbb{R}$ is the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains AP and PC: $PAP = alg_{L^\infty(\mathbb{R})}\{AP, PC\}$.*

Let us use the notation $\mathcal{G}B$ for the group of all invertible elements of a Banach algebra B. The following proposition is the matrix version of a corresponding result for the scalar case (cf. [BKS02, Proposition 3.15]).

Proposition 1. *(a) If $\Gamma \in PAP^{N\times N}$, then there are uniquely determined functions Θ_ℓ, $\Theta_r \in AP^{N\times N}$ and $\Gamma_0 \in PC_0^{N\times N}$ such that*

$$\Gamma = (1-u)\Theta_\ell + u\Theta_r + \Gamma_0, \tag{10.3}$$

where $u \in C(\overline{\mathbb{R}})$, $u(-\infty)=0$, *and* $u(+\infty)=1$.
(b) If $\Gamma \in \mathcal{G}PAP^{N\times N}$, *then there is an invertible semi-almost periodic element* $\Theta \in \mathcal{G}SAP^{N\times N}$ *and an invertible piecewise continuous element* $\Xi \in \mathcal{G}PC^{N\times N}$ *(such that* $\Xi(-\infty) = \Xi(+\infty) = I_{N\times N}$*) which allow the construction of a factorization*

$$\Gamma = \Theta\Xi \tag{10.4}$$

and $W_\Gamma = W_\Theta W_\Xi + K_1 = W_\Xi W_\Theta + K_2$ *with compact operators* $K_1,\ K_2$.

The almost periodic representatives of Θ *are the functions* Θ_ℓ *and* Θ_r *of part (a).*

We will now recall a factorization concept within AP which we will use several times in this chapter.

Definition 2. *A matrix function* $\Gamma \in \mathcal{G}AP^{N\times N}$ *is said to admit a right AP factorization if it can be represented in the form*

$$\Gamma(x) = \Gamma_-(x)D(x)\Gamma_+(x) \tag{10.5}$$

for all $x \in \mathbb{R}$, *with* $\Gamma_- \in \mathcal{G}AP_-^{N\times N}$, $\Gamma_+ \in \mathcal{G}AP_+^{N\times N}$, *and where* D *is a diagonal matrix of the form* $D(x) = \mathrm{diag}[e^{ix\lambda_1},\ldots,e^{ix\lambda_N}]$, $\lambda_j \in \mathbb{R}$. *The numbers* λ_j *are called the right AP indices of the factorization. A right AP factorization with* $D = I_{N\times N}$ *is referred to as a canonical right AP factorization.*

It is said that a matrix function $\Gamma \in \mathcal{G}AP^{N\times N}$ *admits a left AP factorization if instead of* (10.5) *we have* $\Gamma(x) = \Gamma_+(x)D(x)\Gamma_-(x)$ *for all* $x \in \mathbb{R}$ *and* $\Gamma_\pm$ *and* D *having the same property as above.*

From the above definition we can observe that if an invertible almost periodic matrix function Γ admits a right AP factorization, then $\widetilde{\Gamma}$ admits a left AP factorization, and Γ^{-1} also admits a left AP factorization. The vector containing the right AP indices will be denoted by $k(\Gamma)$, i.e., in the above case $k(\Gamma) := (\lambda_1,\ldots,\lambda_N)$. If we consider the case with equal *right AP indices* ($k(\Gamma) := (\lambda_1,\lambda_1,\ldots,\lambda_1)$), then the matrix $\mathbf{d}(\Gamma) := M(\Gamma_-)M(\Gamma_+)$ is independent of the particular choice of the right AP factorization. In this case, this matrix $\mathbf{d}(\Gamma)$ is called the *geometric mean* of Γ.

In order to relate operators and to transfer certain operator properties between the related operators, we will also be using the notion of equivalence after extension for bounded linear operators.

Definition 3. *Consider two bounded linear operators* $T : X_1 \to X_2$ *and* $S : Y_1 \to Y_2$, *acting between Banach spaces. We say that* T *is equivalent after extension to* S *if there are Banach spaces* Z_1 *and* Z_2 *and invertible bounded linear operators* E *and* F *such that*

$$\mathrm{diag}[T, I_{Z_1}] = E\,\mathrm{diag}[S, I_{Z_2}]\,F, \tag{10.6}$$

where I_{Z_1} *and* I_{Z_2} *represent the identity operators in* Z_1 *and* Z_2, *respectively. This relation between* T *and* S *will be denoted by* $T \overset{*}{\sim} S$.

Remark 1. If T is equivalent after extension with S, then T and S have the same Fredholm regularity properties, i.e., one of these operators is invertible, one-sided invertible, Fredholm, semi-Fredholm, one-sided regularizable, generalized invertible, or normally solvable, if and only if the other enjoys that property.

Proposition 2. *Let $\Upsilon \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$. Then $\mathfrak{D}_\Upsilon$ is equivalent after extension to the Wiener–Hopf operator $W_{\Upsilon^{-1}\widetilde{\Upsilon}} : [L^2_+(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^N$ with Fourier symbol $\Upsilon^{-1}\widetilde{\Upsilon}$:*

$$\mathfrak{D}_\Upsilon \overset{*}{\sim} W_{\Upsilon^{-1}\widetilde{\Upsilon}}. \tag{10.7}$$

Proof. We may apply Theorem 2.1 of [CaSi09] to our present case and therefore directly conclude that $\mathfrak{D}_\Upsilon$ is equivalent after extension to the Wiener–Hopf operator $W_\Psi : [L^2_+(\mathbb{R})]^{2N} \to [L^2(\mathbb{R}_+)]^{2N}$ with Fourier symbol:

$$\Psi = \begin{bmatrix} 0 & -I_N \\ \Upsilon^{-1}\widetilde{\Upsilon} & \Upsilon^{-1} \end{bmatrix}.$$

We now observe that this Wiener–Hopf operator W_Ψ is equivalent after extension with the operator $W_{\Upsilon^{-1}\widetilde{\Upsilon}}$. In fact, the following holds:

$$W_\Psi = r_+\mathcal{F}^{-1} \begin{bmatrix} 0 & -I_N \\ I_N & \Upsilon^{-1} \end{bmatrix} \mathcal{F}\ell_0 r_+\mathcal{F}^{-1} \begin{bmatrix} \Upsilon^{-1}\widetilde{\Upsilon} & 0 \\ 0 & I_N \end{bmatrix} \mathcal{F}.$$

This, together with the equivalence after extension relation between $\mathfrak{D}_\Upsilon$ and W_Ψ (and also considering the transitivity of the equivalence after extension relation), leads us to the operator relation (10.7).

10.3 The Fredholm Property

We start by recalling a Fredholm characterization for Wiener–Hopf operators with PAP matrix Fourier symbols having lateral almost periodic representatives admitting right AP factorizations. This result will be used to find sufficient conditions to ensure the Fredholm property of the operators under study.

Theorem 1 (cf., e.g., [BKS02, Theorem 3.16]). *Let $\Gamma \in PAP^{N\times N}$. If $\Gamma \notin \mathcal{G}[PAP]^{N\times N}$, then W_Γ is not semi-Fredholm. Assume now that $\Gamma \in \mathcal{G}PAP$ and Γ_ℓ and Γ_r admit a right AP factorization. Then the Wiener–Hopf operator W_Γ is Fredholm if and only if:*

(i) the almost periodic representatives Γ_ℓ and Γ_r admit canonical right AP factorizations, i.e., with $k(\Gamma_\ell) = k(\Gamma_r) = (0,\ldots,0)$;
(ii) $\mathrm{sp}(\mathbf{d}^{-1}(\Gamma_r)\mathbf{d}(\Gamma_\ell)) \cap (-\infty, 0] = \emptyset$,
(iii) $\mathrm{sp}(\Gamma^{-1}(x-0)\Gamma(x+0)) \cap (-\infty, 0] = \emptyset$

for all $x \in \mathbb{R}$.

From Proposition 1, if $\varUpsilon \in PAP^{N\times N}$ then this matrix function admits the following representation:

$$\varUpsilon = (1-u)\varUpsilon_\ell + u\varUpsilon_r + \varUpsilon_0 \tag{10.8}$$

(with $\varUpsilon_0 \in [PC_0]^{N\times N}$) and so

$$\varUpsilon^{-1}\widetilde{\varUpsilon} = [(1-u)\varUpsilon_\ell + u\varUpsilon_r + \varUpsilon_0]^{-1}[(1-\widetilde{u})\widetilde{\varUpsilon_\ell} + \widetilde{u}\widetilde{\varUpsilon_r} + \widetilde{\varUpsilon_0}]. \tag{10.9}$$

Therefore, from (10.9), we obtain that

$$(\varUpsilon^{-1}\widetilde{\varUpsilon})_\ell = \varUpsilon_\ell^{-1}\widetilde{\varUpsilon_r}, \qquad (\varUpsilon^{-1}\widetilde{\varUpsilon})_r = \varUpsilon_r^{-1}\widetilde{\varUpsilon_\ell}. \tag{10.10}$$

These representations are important not only in the following result but also in the final result where a Fredholm index formula is obtained.

Theorem 2. *Let $\varUpsilon \in \mathcal{G}PAP^{N\times N}$ such that $\varUpsilon_\ell^{-1}\widetilde{\varUpsilon_r}$ admits a right AP factorization. In this case, the operator $\mathfrak{D}_\varUpsilon$ is Fredholm if and only if the following three conditions are satisfied:*

(l) $\varUpsilon_\ell^{-1}\widetilde{\varUpsilon_r}$ admits a canonical right AP factorization;
(ll) $\mathrm{sp}[\mathbf{d}(\varUpsilon_\ell^{-1}\widetilde{\varUpsilon_r})] \cap i\mathbb{R} = \emptyset$*;*
(lll) $\mathrm{sp}[\varUpsilon^{-1}(-x+0)\varUpsilon(x-0)\varUpsilon^{-1}(x+0)\varUpsilon(-x-0)] \cap (-\infty, 0] = \emptyset$*.*

Proof. If $\varUpsilon \in \mathcal{G}[PAP]^{N\times N}$ then $\varUpsilon^{-1}\widetilde{\varUpsilon}$ is also invertible in $PAP^{N\times N}$.

The Fredholm property of $\mathfrak{D}_\varUpsilon$ implies that the operator $W_{\varUpsilon^{-1}\widetilde{\varUpsilon}}$ is also a Fredholm operator (cf. (10.7)). Employing Theorem 1 we obtain that $(\varUpsilon^{-1}\widetilde{\varUpsilon})_\ell$ and $(\varUpsilon^{-1}\widetilde{\varUpsilon})_r$ admit canonical right AP factorizations,

$$\mathrm{sp}[\mathbf{d}^{-1}((\varUpsilon^{-1}\widetilde{\varUpsilon})_r)\mathbf{d}((\varUpsilon^{-1}\widetilde{\varUpsilon})_\ell)] \cap (-\infty, 0] = \emptyset \tag{10.11}$$

and

$$\mathrm{sp}[(\varUpsilon^{-1}\widetilde{\varUpsilon})^{-1}(x-0)(\varUpsilon^{-1}\widetilde{\varUpsilon})(x+0)] \cap (-\infty, 0] = \emptyset. \tag{10.12}$$

Due to (10.10) we conclude that $\varUpsilon_\ell^{-1}\widetilde{\varUpsilon_r}$ admits a canonical right AP factorization and we derive from (10.11) that

$$\mathrm{sp}[\mathbf{d}^{-1}(\varUpsilon_r^{-1}\widetilde{\varUpsilon_\ell})\mathbf{d}(\varUpsilon_\ell^{-1}\widetilde{\varUpsilon_r})] \cap (-\infty, 0] = \emptyset. \tag{10.13}$$

A canonical right AP factorization of $\varUpsilon_\ell^{-1}\widetilde{\varUpsilon_r}$ can be normalized into

$$\varUpsilon_\ell^{-1}\widetilde{\varUpsilon_r} = \varTheta_- \varLambda \varTheta_+, \tag{10.14}$$

where $\varTheta_\pm$ have the same factorization properties as the original lateral factors of the canonical factorization but with $M(\varTheta_\pm) = I$, and where $\varLambda := \mathbf{d}(\varUpsilon_\ell^{-1}\widetilde{\varUpsilon_r})$.

Thus, (10.14) allows $\Upsilon_r^{-1}\widetilde{\Upsilon}_\ell = (\widetilde{\Upsilon_\ell^{-1}\widetilde{\Upsilon}_r})^{-1} = \widetilde{\Theta_+^{-1}}\Lambda^{-1}\widetilde{\Theta_-^{-1}}$, which shows that $\mathbf{d}(\Upsilon_r^{-1}\widetilde{\Upsilon}_\ell) = \Lambda^{-1}$, and therefore (10.13) turns out to be equivalent to $\mathrm{sp}[\Lambda^2] \cap (-\infty, 0] = \emptyset$. From the eigenvalue definition we therefore have the result $\mathrm{sp}[\Lambda] \cap i\mathbb{R} = \emptyset$, which proves condition (ll).

In addition, from (10.12) we derive that

$$\mathrm{sp}[\widetilde{\Upsilon^{-1}}(x-0)\Upsilon(x-0)\Upsilon^{-1}(x+0)\widetilde{\Upsilon}(x+0)] \cap (-\infty, 0] = \emptyset,$$

which is equivalent to

$$\mathrm{sp}[\Upsilon^{-1}(-x+0)\Upsilon(x-0)\Upsilon^{-1}(x+0)\Upsilon(-x-0)] \cap (-\infty, 0] = \emptyset.$$

Let us now assume that conditions (l)–(lll) hold and prove that $\mathfrak{D}_\Upsilon$ is a Fredholm operator. Since $\Upsilon_\ell^{-1}\widetilde{\Upsilon_r} = (\Upsilon^{-1}\widetilde{\Upsilon})_\ell$ admits a canonical right AP factorization, then $(\widetilde{\Upsilon^{-1}\widetilde{\Upsilon}})_\ell = \widetilde{\Upsilon_\ell^{-1}}\Upsilon_r$ admits a canonical left AP factorization and $[(\widetilde{\Upsilon^{-1}\widetilde{\Upsilon}})_\ell]^{-1} = \Upsilon_r^{-1}\widetilde{\Upsilon}_\ell$ admits a canonical right AP factorization. These last two canonical right AP factorizations and condition (ll) imply that $\mathrm{sp}[\mathbf{d}^{-1}((\Upsilon^{-1}\widetilde{\Upsilon})_r)\mathbf{d}((\Upsilon^{-1}\widetilde{\Upsilon})_\ell)] \cap (-\infty, 0] = \mathrm{sp}[\mathbf{d}^{-1}(\Upsilon_r^{-1}\widetilde{\Upsilon}_\ell)\mathbf{d}(\Upsilon_\ell^{-1}\widetilde{\Upsilon_r})] \cap (-\infty, 0] = \emptyset$.

Condition (lll) allows us to conclude that

$$\mathrm{sp}[(\Upsilon^{-1}\widetilde{\Upsilon})^{-1}(x-0)(\Upsilon^{-1}\widetilde{\Upsilon})(x+0)] \cap (-\infty, 0] = \emptyset.$$

All these facts together and Theorem 1 show that $W_{\Upsilon^{-1}\widetilde{\Upsilon}}$ is a Fredholm operator. Using the equivalence after extension relation presented in Proposition 2, we obtain that $\mathfrak{D}_\Upsilon$ is a Fredholm operator.

10.4 The Fredholm Index Formula

In this section we will concentrate on obtaining a Fredholm index formula for $\mathfrak{D}_\Upsilon$, i.e., for the sum of Wiener–Hopf plus and minus Hankel operators $W_{\widetilde{\Upsilon}} \pm H_\Upsilon$ with piecewise almost periodic Fourier symbols such that $\Upsilon_\ell^{-1}\widetilde{\Upsilon_r}$ admits a right AP factorization. For that purpose, taking into account that $PAP = SAP + PC_0$, we will first recall some known properties of Wiener–Hopf plus Hankel operators with symbols in SAP and with symbols in PC. Within this context, let us assume that $W_{\widetilde{\Upsilon}} + H_\Upsilon$ and $W_{\widetilde{\Upsilon}} - H_\Upsilon$ have the Fredholm property.

Let $\mathcal{G}SAP_{0,0}$ denote the set of all functions $\varphi \in \mathcal{G}SAP$ for which $k(\varphi_\ell) = k(\varphi_r) = 0$. To define the Cauchy index of $\varphi \in \mathcal{G}SAP_{0,0}$ we need the next lemma.

Lemma 1 ([BKS02, Lemma 3.12]). *Let $A \subset (0, \infty)$ be an unbounded set and let $\{I_\alpha\}_{\alpha\in A} = \{(x_\alpha, y_\alpha)\}_{\alpha\in A}$ be a family of intervals such that $x_\alpha \geq 0$*

and $|I_\alpha| = y_\alpha - x_\alpha \to \infty$*, as* $\alpha \to \infty$*. If* $\varphi \in \mathcal{G}SAP_{0,0}$ *and* $\arg\varphi$ *is any continuous argument of* φ*, then the limit*

$$\frac{1}{2\pi} \lim_{\alpha\to\infty} \frac{1}{|I_\alpha|} \int_{I_\alpha} ((\arg\varphi)(x) - (\arg\varphi)(-x))dx \tag{10.15}$$

exists, is finite, and is independent of the particular choices of $\{(x_\alpha, y_\alpha)\}_{\alpha\in A}$ *and* $\arg\varphi$*.*

The limit (10.15) is denoted by ind φ and is usually called the *Cauchy index* of φ.

The following theorem provides a formula for the Fredholm index of matrix Wiener–Hopf operators with SAP Fourier symbols.

Theorem 3 ([BKS02, Theorem 10.12]). *Let* $\Gamma \in SAP^{N\times N}$*. If the almost periodic representatives* Γ_ℓ, Γ_r *admit right AP factorizations, and if* W_Γ *is a Fredholm operator, then*

$$\text{Ind } W_\Gamma = -\text{ind}[\det \Gamma] - \sum_{k=1}^{N} \Big(\frac{1}{2} - \Big\{\frac{1}{2} - \frac{1}{2\pi}\arg\,\xi_k\Big\}\Big), \tag{10.16}$$

where $\xi_1, \dots, \xi_N \in \mathbb{C}\backslash(-\infty, 0]$ *are the eigenvalues of the matrix* $\mathbf{d}^{-1}(\Gamma_r)\mathbf{d}(\Gamma_\ell)$ *and* $\{\cdot\}$ *stands for the fractional part of a real number. Additionally, when choosing* $\arg\,\xi_k$ *in* $(-\pi, \pi)$*, we have*

$$\text{Ind } W_\Gamma = -\text{ind}[\det \Gamma] - \frac{1}{2\pi}\sum_{k=1}^{N} \arg\ \xi_k.$$

Let us now consider $\Gamma \in PC^{N\times N}$. Recall the auxiliary extension of Γ defined by

$$\Gamma^{\#}(x,\mu) := (1-\mu)\Gamma(x-0) + \mu\Gamma(x+0), \qquad (x,\mu) \in \dot{\mathbb{R}} \times [0,1]$$

and consider $\det \Gamma^{\#} : \dot{\mathbb{R}} \times [0,1] \to \mathbb{C}$. The set

$$C := \{\det \Gamma^{\#}(x,\mu) \in \mathbb{C} : x \in \mathbb{R}, \mu \in [0,1]\}$$

is a closed continuous curve obtained from Γ by joining $\det \Gamma^{\#}(x-0)$ to $\det \Gamma^{\#}(x+0)$ through a line segment at the discontinuity points of Γ. If $0 \notin C$, then the *winding number* of C with respect to the origin (denoted in this case by $\text{wind}[\det \Gamma^{\#}]$) is defined as the counter-clockwise circuits around the origin performed by the image of $\det \Gamma^{\#}$. Suppose $\det \Gamma^{\#}(x,\mu) \neq 0$ for all $(x,\mu) \in \mathbb{R} \times [0,1]$ which implies that $\Gamma(x-0)$ and $\Gamma(x+0)$ are invertible for all $x \in \dot{\mathbb{R}}$. Moreover, assume that the set $\Delta_\Gamma := \{x \in \dot{\mathbb{R}} : \Gamma(x-0) \neq \Gamma(x+0)\}$ is finite. For a connected component ℓ of $\dot{\mathbb{R}}\backslash\Delta_\Gamma$ we define $\text{ind}_\ell\Gamma$ as $(2\pi)^{-1}$ times the increment of any continuous argument of $\det \Gamma$ on ℓ. Taking into account the possible jump at infinity, the winding number of C can be given by:

$$\mathrm{wind}[\det \Gamma^{\#}] = \mathrm{ind}[\det \Gamma^{\#}] + \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \mathrm{arg} \xi_k(\infty) \right\} \right), \tag{10.17}$$

where

$$\mathrm{ind}[\det \Gamma^{\#}] = \sum_{\ell} \mathrm{ind}_{\ell}[\det \Gamma] + \sum_{x \in \Delta_\Gamma} \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \mathrm{arg} \xi_k(x) \right\} \right), \tag{10.18}$$

with $\xi_1(x), \ldots, \xi_N(x)$ being the eigenvalues of $\Gamma^{-1}(x-0)\Gamma(x+0)$ for $x \in \Delta_\Gamma$ and $\{\cdot\}$ denoting the fractional part of a real number.

Theorem 4 (cf., e.g., [BKS02, Theorem 5.10]). *Let $\Gamma \in PC^{N\times N}$. If W_Γ is a Fredholm operator and Γ has at most finitely many jumps, then*

$$\mathrm{Ind} W_\Gamma = -\mathrm{wind}(\det \Gamma^{\#})$$

where wind$(\det \Gamma^{\#})$ *is given by* (10.17)–(10.18). *Choosing the arguments in* $(-\pi, \pi)$, *we also have*

$$\mathrm{Ind} W_\Gamma = -\mathrm{ind}[\det\ \Gamma^{\#}] - \frac{1}{2\pi} \sum_{k=1}^{N} \mathrm{arg} \xi_k(\infty)$$

Now, we are in conditions to conclude a formula for the Fredholm index of matrix Wiener–Hopf operators with PAP Fourier symbols.

Let $\Gamma \in PAP^{N\times N}$. Then $\Gamma = \Theta\Xi$ (with $\Theta \in \mathcal{G}SAP^{N\times N}$, $\Xi \in \mathcal{G}PC^{N\times N}$ and $\Xi(\pm\infty) = I_{N\times N}$) such that

$$W_\Gamma = W_\Theta W_\Xi + K \tag{10.19}$$

with K being a compact operator (cf. Proposition 1). Assume that W_Γ is a Fredholm operator. From (10.19) we derive that

$$\mathrm{Ind} W_\Gamma = \mathrm{Ind} W_\Theta + \mathrm{Ind} W_\Xi$$

Using now formulas (10.16), (10.17), and (10.18) and taking into account that Ξ does not have a jump at infinity, we can conclude the following theorem:

Theorem 5 (cf. [BoCa, Proposition 6.3]). *Let $\Gamma \in \mathcal{G}PAP^{N\times N}$. If the almost periodic representatives Γ_ℓ, Γ_r admit right AP factorizations, and if W_Γ is a Fredholm operator, then*

$$\mathrm{Ind}\ W_\Gamma = -\sum_{\ell} \mathrm{ind}_{\ell}[\det\ \Xi] - \mathrm{ind}[\det \Theta]$$
$$- \sum_{x \in \Delta_\Gamma} \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \mathrm{arg}\, \xi_k(x) \right\} \right) - \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \mathrm{arg}\, \eta_k \right\} \right), \tag{10.20}$$

where $\xi_k(x)$ *are the eigenvalues of the matrix function* $\Gamma^{-1}(x-0)\Gamma(x+0)$ *and* η_k *are the eigenvalues of the matrix* $\mathbf{d}^{-1}(\Gamma_r)\mathbf{d}(\Gamma_\ell)$*. Choosing both arguments in* $(-\pi,\pi)$, (10.20) *simplifies to*

$$\mathrm{Ind} W_\Gamma = -\sum_\ell \mathrm{ind}_\ell[\det\ \Xi] - \mathrm{ind}[\det\Theta] - \frac{1}{2\pi}\sum_{x\in\Delta_\Gamma}\sum_{k=1}^{N}\arg\xi_k(x) - \frac{1}{2\pi}\sum_{k=1}^{N}\arg\eta_k. \tag{10.21}$$

We have now a full machinery to obtain a Fredholm index formula for the operator $\mathfrak{D}_\Upsilon$ (cf. (10.1)).

Theorem 6. *Let* $\Upsilon\in\mathcal{G}PAP^{N\times N}$ *such that* $\Upsilon_\ell^{-1}\widetilde{\Upsilon_r}$ *admits a right AP factorization. If* $W_{\widetilde{\Upsilon}}\pm H_\Upsilon$ *are Fredholm operators, then*

$$\mathrm{Ind}[W_{\widetilde{\Upsilon}}+H_\Upsilon]+\mathrm{Ind}[W_{\widetilde{\Upsilon}}-H_\Upsilon] = -\sum_\ell \mathrm{ind}_\ell[\det\Xi] - \mathrm{ind}[\det\ \Theta]$$
$$-\sum_{x\in\Delta_{\Upsilon^{-1}\widetilde{\Upsilon}}}\sum_{k=1}^{N}\left(\frac{1}{2}-\left\{\frac{1}{2}-\frac{1}{2\pi}\arg\xi_k(x)\right\}\right)-\sum_{k=1}^{N}\left(\frac{1}{2}-\left\{\frac{1}{2}-\frac{1}{\pi}\arg\delta_k\right\}\right), \tag{10.22}$$

where $\Upsilon^{-1}\widetilde{\Upsilon}=\Theta\Xi$ *is a corresponding factorization in the sense of* (10.4) *for the invertible matrix-valued PAP function* $\Upsilon^{-1}\widetilde{\Upsilon}$, $\xi_k(x)$ *are the eigenvalues of the matrix function* $\Upsilon^{-1}(-x+0)\Upsilon(x-0)\Upsilon^{-1}(x+0)\Upsilon(-x-0)$, $\delta_k\in\mathbb{C}\backslash i\mathbb{R}$ *are the eigenvalues of the matrix* $\mathbf{d}(\Upsilon_\ell^{-1}\widetilde{\Upsilon_r})$ *and*

$$\Delta_{\Upsilon^{-1}\widetilde{\Upsilon}}=\left\{x\in\dot{\mathbb{R}}:\left(\Upsilon^{-1}\widetilde{\Upsilon}\right)(x-0)\neq\left(\Upsilon^{-1}\widetilde{\Upsilon}\right)(x+0)\right\}.$$

Moreover, formula (10.22) *simplifies into the following one:*

$$\mathrm{Ind}[W_{\widetilde{\Upsilon}}+H_\Upsilon]+\mathrm{Ind}[W_{\widetilde{\Upsilon}}-H_\Upsilon] = -\sum_\ell \mathrm{ind}_\ell[\det\Xi] - \mathrm{ind}[\det\Theta]$$
$$-\frac{1}{2\pi}\sum_{x\in\Delta_{\Upsilon^{-1}\widetilde{\Upsilon}}}\sum_{k=1}^{N}\arg\xi_k(x)-\frac{1}{\pi}\sum_{k=1}^{N}\arg\beta(\delta_k) \tag{10.23}$$

when choosing $\arg\xi_k(x)\in(-\pi,\pi)$ *and*

$$\beta(\delta_k)=\begin{cases}\arg(\delta_k) & \text{if } \Re e\ \delta_k>0\\ \arg(-\delta_k) & \text{if } \Re e\ \delta_k<0\end{cases}$$

with the argument in both cases being taken in $(-\frac{\pi}{2},\frac{\pi}{2})$.

Proof. Using the equivalence after extension relation stated in Proposition 2, we conclude that $\mathrm{Ind}\ \mathfrak{D}_\Upsilon=\mathrm{Ind}\ W_{\Upsilon^{-1}\widetilde{\Upsilon}}$ and consequently, $\mathrm{Ind}[W_{\widetilde{\Upsilon}}+H_\Upsilon]+\mathrm{Ind}[W_{\widetilde{\Upsilon}}-H_\Upsilon]=\mathrm{Ind}\ W_{\Upsilon^{-1}\widetilde{\Upsilon}}$. Following (10.20), we obtain

$$\mathrm{Ind} W_{\Upsilon^{-1}\widetilde{\Upsilon}} = -\sum_{\ell} \mathrm{ind}_\ell[\det \Xi] - \mathrm{ind}[\det \Theta]$$
$$-\sum_{x\in\Delta_{\Upsilon^{-1}\widetilde{\Upsilon}}} \sum_{k=1}^{N} \Big(\frac{1}{2} - \Big\{\frac{1}{2} - \frac{1}{2\pi}\arg\xi_k(x)\Big\}\Big) - \sum_{k=1}^{N} \Big(\frac{1}{2} - \Big\{\frac{1}{2} - \frac{1}{2\pi}\arg\zeta_k\Big\}\Big), \quad (10.24)$$

where $\Upsilon^{-1}\widetilde{\Upsilon} = \Theta\Xi$ is a corresponding factorization in the sense of (10.3) for the invertible PAP matrix function $\Upsilon^{-1}\widetilde{\Upsilon}$, $\xi_k(x)$ are the eigenvalues of the matrix function $\Upsilon^{-1}(-x+0)\Upsilon(x-0)\Upsilon^{-1}(x+0)\Upsilon(-x-0)$ and $\zeta_k \in \mathbb{C}\backslash(-\infty, 0]$ are the eigenvalues of the matrix $\mathbf{d}^{-1}((\Upsilon^{-1}\widetilde{\Upsilon})_r)\mathbf{d}((\Upsilon^{-1}\widetilde{\Upsilon})_\ell)$. However, since we have already proved that $\mathbf{d}^{-1}((\Upsilon^{-1}\widetilde{\Upsilon})_r)\mathbf{d}((\Upsilon^{-1}\widetilde{\Upsilon})_\ell) = \Lambda^2$, it turns out that formula (10.24) simplifies into (10.22).

Finally, the simplification into the formula (10.23) occurs as a direct consequence of the above indicated choice of arguments for the eigenvalues.

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Invertibility of Singular Integral Operators with Flip Through Explicit Operator Relations

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11.1 Introduction

The integral equations which are characterized by singular integral operators with shift appear frequently in a large variety of applied problems (we refer to [KaSa01, KrLi94] for a general background on these operators and historical references). Thus, it is of fundamental importance to obtain descriptions of the invertibility characteristics of these operators. Although some invertibility criteria are presently known for several classes of singular integral operators with shift, the corresponding criteria still remain to be achieved for many others. In addition, among all the classes of singular integral operators with shifts, the ones with weighted shifts typically reveal extra difficulties.

This chapter is devoted to the analysis of singular integral operators

$$\mathcal{A} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 \widetilde{J} + b_1 S_{\mathbb{T}} \widetilde{J}, \tag{11.1}$$

with essentially bounded functions a_0, b_0, a_1, b_1 and a *flip operator* $\widetilde{J}$ which contains a backward Carleman shift α (or, more precisely, a weighted backward Carleman shift) in the form $(\widetilde{J}\varphi)(t) = \frac{1}{t}\varphi\left(\frac{1}{t}\right)$, $t \in \mathbb{T}$, and that is defined between weighted Lebesgue spaces $L^p(\mathbb{T}, \rho)$, $1 < p < \infty$, $\rho(t) = |t-1|^{1-2/p}$. The operator $I_{\mathbb{T}}$ denotes the identity operator, and $S_{\mathbb{T}}$ denotes the Cauchy singular integral operator along the unit circle $\mathbb{T}$ being defined almost everywhere by

$$(S_{\mathbb{T}} f)(t) = \frac{1}{\pi i} \int_{\mathbb{T}} \frac{f(\tau)}{\tau - t} d\tau$$

(where the integral is understood in the sense of principal value).

Explicit operator equivalence relations will be exhibited between (11.1) and singular integral operators without shifts. As a consequence, it will be possible to present a new matrix Toeplitz operator that will be preponderant in determining an invertibility criterion for the initial operators. Due to the explicit form of the present operator relations, in the case of invertibility, this

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technique allows us to find formulas to the inverse of $\mathcal{A}$. Moreover, due to the *generalized factorization* of a bounded measurable matrix-valued function (and by using the properties of the factors in such a factorization), formulas for the left-sided and right-sided inverses of the initial operator will also be obtained. In this sense, this chapter may be viewed as a natural continuation of [CaRo09]. For other recent works in this direction, see [CaRo08a, CaRo08b, Ka04].

11.2 Notation and Auxiliary Results

To fix additional notation, let us mention that the weighted Lebesgue space over G, $L^p(G,w)$, is equipped with the norm $\|f\|_{p,w} := \|wf\|_p$, where $\|\cdot\|_p$ denotes the usual norm of $L^p(G)$. In addition, $\mathcal{L}(X,Y)$ will denote the space of all bounded and linear operators defined from the Banach space X into the Banach space Y, and in order to shorten the notation we will also use $\mathcal{L}(X) := \mathcal{L}(X,X)$.

We recall that two bounded and linear operators $T : X_1 \to X_2$ and $S : Y_1 \to Y_2$, acting between Banach spaces are said to be *equivalent* [BaTs92, CaSp98] if there are two boundedly invertible linear operators, $E : Y_2 \to X_2$ and $F : X_1 \to Y_1$, such that $T = E\ S\ F$. We will refer to the last formula as an *operator equivalence relation* (between T and S). In the particular case of $E = F^{-1}$, we say that we have a *similarity relation* between the operators T and S. In the sequel of the work we will use the notion of *equivalence after extension relation* (cf., e.g., [BaTs92]): the operators T and S are called *equivalent after extension* if Banach spaces Z and W exist such that $T \oplus I_Z$ and $S \oplus I_W$ are equivalent operators.

We also recall that a normally solvable operator $T : X \to Y$ (acting between Banach spaces) is called a *Fredholm operator* if $n(t) := \dim\ker T < \infty$ and $d(t) := \dim X/\mathrm{Im}T < \infty$. The *Fredholm index* is then defined by $\mathrm{Ind}\, T := n(T) - d(T)$.

For two equivalent operators (or equivalent after extension) T and S, it follows that T is invertible or has the Fredholm property if and only if S is invertible or has the Fredholm property, respectively.

11.3 Similarity Relations for Singular Integral Operators

In the first part of this section, we will describe the operator relations between operators of the form (11.1) and corresponding matrix operators without shift derived in [CaRo09]. This also allows a relation between $\mathcal{A}$ and a matrix Toeplitz operator which will be useful for describing the inverse formulas for operator $\mathcal{A}$.

A first step in this strategy is to apply the isometric isomorphism $B : L^p(\mathbb{T},\rho) \to L^p(\mathbb{R})$ (with $\rho(t) = |t-1|^{1-2/p}$, $t \in \mathbb{T}$), defined by

$$(B\phi)(x) = \frac{2^{1-1/p}}{x+i}\phi\left(\frac{x-i}{x+i}\right), \quad x \in \mathbb{R} \tag{11.2}$$

to operator $\mathcal{A}$ in a way such that we will reach the new singular integral operator:

$$\mathcal{B} = B\mathcal{A}B^{-1} = aI_{\mathbb{R}} + bS_{\mathbb{R}} + cW_{\mathbb{R}} + dS_{\mathbb{R}}W_{\mathbb{R}} : L^p(\mathbb{R}) \to L^p(\mathbb{R}), \tag{11.3}$$

where $S_{\mathbb{R}}$ is the Cauchy singular integral operator over $\mathbb{R}$ defined by

$$(S_{\mathbb{R}}\varphi)(x) = \frac{1}{\pi i}\int_{\mathbb{R}} \frac{f(u)}{u-x}du,$$

$x \in \mathbb{R}$ (the integral being considered in the principal value sense), and $W_{\mathbb{R}}$ is called the *reflection operator* on $\mathbb{R}$ defined by $(W_{\mathbb{R}}f)(x) = f(-x)$, $x \in \mathbb{R}$. In addition, we have $BaB^{-1} = (B_0^{-1}a)I_{\mathbb{R}}$, where for $a \in L^\infty(\mathbb{T})$ the operator B_0^{-1} is given by $(B_0^{-1}a)(x) = a(\frac{x-i}{x+i})$, $x \in \mathbb{R}$, and $(B_0a)(t) = a(i\frac{1+t}{1-t})$, $t \in \mathbb{T} \setminus \{1\}$.

Thus, the coefficients of operator $\mathcal{B}$ are defined by $a = B_0^{-1}a_0$, $b = B_0^{-1}b_0$, $c = -B_0^{-1}a_1$, $d = -B_0^{-1}b_1$. In addition, due to [Ka01], we know that it is possible to construct an operator equivalence relation between the singular integral operator $\mathcal{B}$ given in (11.3) and a pure singular integral operator

$$D_{\mathbb{R}_+} := \mathcal{H}\mathcal{B}\mathcal{F} = u_{\mathbb{R}_+}I_{\mathbb{R}_+} + v_{\mathbb{R}_+}S_{\mathbb{R}_+} \quad \in \quad \mathcal{L}([L^p(\mathbb{R}_+, |x|^{-1/2p})]^2), \tag{11.4}$$

where $\mathcal{H} = N_{\mathbb{R}_+}^{-1}K^{-1}M_{\mathbb{R}_+}^{-1}$ and $\mathcal{F} = M_{\mathbb{R}_+}KR_{\mathbb{R}_+}N_{\mathbb{R}_+}$ are constructed based on the invertible operator $M_{\mathbb{R}_+} \in \mathcal{L}([L^p(\mathbb{R}_+)]^2, L^p(\mathbb{R}))$ defined by

$$M_{\mathbb{R}_+}\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = \varphi(x) := \begin{cases} \varphi_1(x), & x \in \mathbb{R}_+ \\ \varphi_2(-x), & x \in \mathbb{R}_- \end{cases} \tag{11.5}$$

(where $\mathbb{R}_+ := (0 + \infty)$ and $\mathbb{R}_- := (-\infty, 0)$), the idempotent operator

$$K^{\pm 1} = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I \end{pmatrix} \in \mathcal{L}([L^p(\mathbb{R}_+)]^2), \tag{11.6}$$

the invertible *squaring variable* and *cutting weight* operator $N_{\mathbb{R}_+}$ defined by

$$(N_{\mathbb{R}_+}\varphi)(x) = \varphi(x^2), \qquad N_{\mathbb{R}_+} \in \mathcal{L}([L^p(\mathbb{R}_+, |x|^{-1/2p})]^2, [L^p(\mathbb{R}_+)]^2), \tag{11.7}$$

and the operator $R_{\mathbb{R}_+}$ given by

$$R_{\mathbb{R}_+} = \begin{pmatrix} S_{\mathbb{R}_+} + U_{1,\mathbb{R}_+} & 0 \\ 0 & I_{\mathbb{R}_+} \end{pmatrix} \in \mathcal{L}([L^p(\mathbb{R}_+)]^2), \tag{11.8}$$

where $(S_{\mathbb{R}_+}f)(x) = \frac{1}{\pi i}\int_{\mathbb{R}_+}\frac{f(u)}{u-x}du$ and $(U_{1,\mathbb{R}_+}f)(x) = \frac{1}{\pi i}\int_{\mathbb{R}_+}\frac{f(u)}{u+x}du$, $x \in \mathbb{R}_+$. Note that $S_{\mathbb{R}_+} + U_{1,\mathbb{R}_+}$ is an invertible operator and its inverse is given by $S_{\mathbb{R}_+} - U_{1,\mathbb{R}_+}$. Thus, $R_{\mathbb{R}_+}$ is also an invertible operator.

In addition, the relation between the coefficients $u_{\mathbb{R}_+}$ and $v_{\mathbb{R}_+}$ of the new operator $D_{\mathbb{R}_+}$ in (11.4) and the coefficients of the operator $\mathcal{A}$ is given by the formulae:

$$u_{\mathbb{R}_+}(x) = \frac{1}{2}\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}, \qquad v_{\mathbb{R}_+}(x) = \frac{1}{2}\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}, \tag{11.9}$$

where $u_{11} = a_1(y)+b_1(y)-(a_1(-y)+b_1(-y))$, $u_{12} = a_0(y)+b_0(y)-(a_0(-y)+b_0(-y))$, $u_{21} = a_1(y) + b_1(y) + (a_1(-y) + b_1(-y))$, $u_{22} = a_0(y) + b_0(y) + (a_0(-y) + b_0(-y))$, $v_{11} = a_0(y) - b_0(y) + a_0(-y) - b_0(-y)$, $v_{12} = a_1(y) - b_1(y) + a_1(-y) - b_1(-y)$, $v_{21} = a_0(y) - b_0(y) - (a_0(-y) - b_0(-y))$, $v_{22} = a_1(y)-b_1(y)-(a_1(-y)-b_1(-y))$, and here we are using the change of variable $y = (x^{1/2} - i)/(x^{1/2} + i)$, $x \in \mathbb{R}_+$.

All the above presented operator identities allow the identification of the operator relation stated in the next result.

Theorem 1. *The singular integral operator* $\mathcal{A}$ (*with backward Carleman shift* $(\widetilde{J}\varphi)(t) = \frac{1}{t}\varphi\left(\frac{1}{t}\right)$, $t \in \mathbb{T}$) *defined by* $\mathcal{A} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 \widetilde{J} + b_1 S_{\mathbb{T}}\widetilde{J}$ *and acting between the space* $L^p(\mathbb{T}, \rho)$ (*with* $\rho(t) = |t-1|^{1-2/p}$) *is equivalent to the matrix* (*pure*) *singular integral operator*

$$D_{\mathbb{R}_+} = u_{\mathbb{R}_+} I_{\mathbb{R}_+} + v_{\mathbb{R}_+} S_{\mathbb{R}_+} \quad \in \quad \mathcal{L}[L^p(\mathbb{R}_+, |x|^{-1/2p})]^2. \tag{11.10}$$

The equivalence relation has the explicit form $\mathcal{G}\mathcal{A}\mathcal{V} = D_{\mathbb{R}_+}$, *where*

$$\begin{aligned} \mathcal{G} &= N_{\mathbb{R}_+}^{-1} K^{-1} M_{\mathbb{R}_+}^{-1} B \;\in\; \mathcal{L}(L^p(\mathbb{T},\rho), [L^p(\mathbb{R}_+, |x|^{-1/2p})]^2), \\ \mathcal{V} &= B^{-1} M_{\mathbb{R}_+} K R_{\mathbb{R}_+} N_{\mathbb{R}_+} \;\in\; \mathcal{L}([L^p(\mathbb{R}_+, |x|^{-1/2p})]^2, L^p(\mathbb{T},\rho)). \end{aligned}$$

On the other hand, we will extend the operator $D_{\mathbb{R}_+}$ in (11.10) by the identity into the whole $[L^p(\mathbb{R}, |x|^{-1/2p})]^2$ space. The resulting operator from this equivalence after extension relation applied to $D_{\mathbb{R}_+}$ has the form

$$D_{\mathbb{R}} := \begin{pmatrix} D_{\mathbb{R}_+} & 0 \\ 0 & I_{[L^p(\mathbb{R}_-, |x|^{-1/2p})]^2} \end{pmatrix} \in \mathcal{L}([L^p(\mathbb{R}_+, |x|^{-1/2p})]^2 \oplus [L^p(\mathbb{R}_-, |x|^{-1/2p})]^2). \tag{11.11}$$

Thus, $D_{\mathbb{R}_+} : [L^p(\mathbb{R}_+, |x|^{-1/2p})]^2 \to [L^p(\mathbb{R}_+, |x|^{-1/2p})]^2$ can be viewed as the restriction of $D_{\mathbb{R}}$ to its first component spaces. We will use the following notation for this interpretation:

$$D_{\mathbb{R}_+} = \mathrm{Rest}_{[L^p(\mathbb{R}_+, |x|^{-1/2p})]^2}(D_{\mathbb{R}}).$$

It directly follows from identity (11.11) that $D_{\mathbb{R}_+}$ and $D_{\mathbb{R}}$ enjoy the same Fredholm as well as invertibility properties. In addition, the operator $D_{\mathbb{R}}$ can also be written in the form $D_{\mathbb{R}} = u_{\mathbb{R}} I_{\mathbb{R}} + v_{\mathbb{R}} S_{\mathbb{R}}$, where $u_{\mathbb{R}} = \chi_{\mathbb{R}_-} + \ell_0 u_{\mathbb{R}_+}$, $v_{\mathbb{R}} = \ell_0 v_{\mathbb{R}_+}$, where ℓ_0 is the zero extension operator, and $\chi_{\mathbb{R}_-}$ is the characteristic function on $\mathbb{R}_-$. We will now pass from $\mathcal{D}_{\mathbb{R}}$ to a singular integral operator

$\mathcal{D}_{\mathbb{T}}$ defined on the unit circle $\mathbb{T}$ by means of the isometric isomorphism $B_2 := \mathrm{diag}(B, B)$ from $[L^p(\mathbb{R}, |x|^{-1/2p})]^2$ onto $[L^p(\mathbb{T}, w)]^2$ with the weight $w(t) = |i\frac{1+t}{1-t}|^{-1/2p}|1-t|^{1-2/p}$. Therefore, we obtain in explicit form:

$$\mathcal{D}_{\mathbb{T}} := B_2^{-1}\mathcal{D}_{\mathbb{R}}B_2 = u_{\mathbb{T}}I_{\mathbb{T}} + v_{\mathbb{T}}S_{\mathbb{T}}. \tag{11.12}$$

Note that the form of the weight w is a consequence of the use of the operator B_2 (see for instance, [BKS02]), with

$$u_{\mathbb{T}}I_{\mathbb{T}} = B_2^{-1}u_{\mathbb{R}}B_2, \qquad v_{\mathbb{T}}I_{\mathbb{T}} = B_2^{-1}v_{\mathbb{R}}B_2, \tag{11.13}$$

where $u_{\mathbb{T}} = \mathrm{diag}(B_0, B_0)u_{\mathbb{R}}$ and $v_{\mathbb{T}} = \mathrm{diag}(B_0, B_0)v_{\mathbb{R}}$ in $\mathbb{T}_+$, and $u_{\mathbb{T}} \equiv I_{2\times 2}$, $v_{\mathbb{T}} \equiv 0_{2\times 2}$ in $\mathbb{T}_-$. The explicit form of these matrix functions is given by

$$u_{\mathbb{T}}(t) = \begin{cases} u_{\mathbb{T}_+}(t), & t \in \mathbb{T}_+ \\ I_{2\times 2}, & t \in \mathbb{T}_- \end{cases}, \qquad v_{\mathbb{T}}(t) = \begin{cases} v_{\mathbb{T}_+}(t), & t \in \mathbb{T}_+ \\ 0_{2\times 2}, & t \in \mathbb{T}_- \end{cases}, \tag{11.14}$$

where for $t \in \mathbb{T}_+$ we have

$$u_{\mathbb{T}_+}(t) = \frac{1}{2}\begin{pmatrix} \mu_{11}(t) & \mu_{12}(t) \\ \mu_{21}(t) & \mu_{22}(t) \end{pmatrix} \tag{11.15}$$

with

$$\begin{aligned}
\mu_{11}(t) &= (a_1(t^{1/2}) + b_1(t^{1/2})) - (a_1(-t^{1/2}) + b_1(-t^{1/2})) \\
\mu_{12}(t) &= (a_0(t^{1/2}) + b_0(t^{1/2})) - (a_0(-t^{1/2}) + b_0(-t^{1/2})) \\
\mu_{21}(t) &= (a_1(t^{1/2}) + b_1(t^{1/2})) + (a_1(-t^{1/2}) + b_1(-t^{1/2})) \\
\mu_{22}(t) &= (a_0(t^{1/2}) + b_0(t^{1/2})) + (a_0(-t^{1/2}) + b_0(-t^{1/2}))
\end{aligned}$$

and

$$v_{\mathbb{T}_+}(t) = \frac{1}{2}\begin{pmatrix} \vartheta_{11}(t) & \vartheta_{12}(t) \\ \vartheta_{21}(t) & \vartheta_{22}(t) \end{pmatrix} \tag{11.16}$$

with

$$\begin{aligned}
\vartheta_{11}(t) &= (a_0(t^{1/2}) - b_0(t^{1/2})) + (a_0(-t^{1/2}) - b_0(-t^{1/2})) \\
\vartheta_{12}(t) &= (a_1(t^{1/2}) - b_1(t^{1/2})) + (a_1(-t^{1/2}) - b_1(-t^{1/2})) \\
\vartheta_{21}(t) &= (a_0(t^{1/2}) - b_0(t^{1/2})) - (a_0(-t^{1/2}) - b_0(-t^{1/2})) \\
\vartheta_{22}(t) &= (a_1(t^{1/2}) - b_1(t^{1/2})) - (a_1(-t^{1/2}) - b_1(-t^{1/2})).
\end{aligned}$$

In the next theorem we assemble all the above operator relations.

Theorem 2. *The singular integral operator $\mathcal{A}$, with backward Carleman shift $(\widetilde{J}\varphi)(t) = \frac{1}{t}\varphi\left(\frac{1}{t}\right)$, $t \in \mathbb{T}$, defined by $\mathcal{A} = a_0I_{\mathbb{T}} + b_0S_{\mathbb{T}} + a_1\widetilde{J} + b_1S_{\mathbb{T}}\widetilde{J}$ and acting on the space $L^p(\mathbb{T}, \rho)$, $\rho(t) = |t-1|^{1-2/p}$, is equivalent after extension to the matrix singular integral operator $\mathcal{D}_{\mathbb{T}} = u_{\mathbb{T}}I_{\mathbb{T}} + v_{\mathbb{T}}S_{\mathbb{T}} \in \mathcal{L}[L^p(\mathbb{T}, w)]^2$, where $w(t) = |i\frac{1+t}{1-t}|^{-1/2p}|1-t|^{1-2/p}$, and with coefficients $u_{\mathbb{T}}$ and $v_{\mathbb{T}}$ given by* (11.14), (11.15), *and* (11.16).

Now we will relate the operator $\mathcal{A}$ defined on (11.1) with a matrix Toeplitz operator through an additional operator relation. In view of this, in the following result $P_{\mathbb{T}}$ stands for the Riesz projection in $[L^p(\mathbb{T},w)]^2$: $P_{\mathbb{T}} = \frac{1}{2}(I_{\mathbb{T}} + S_{\mathbb{T}})$.

Corollary 1. *For each $t \in \mathbb{T}$, assume that $a_0(t) \neq b_0(t)$ or $a_1(t) \neq b_1(t)$. The singular integral operator with shift $\mathcal{A} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 \widetilde{J} + b_1 S_{\mathbb{T}} \widetilde{J}$ is equivalent to the matrix Toeplitz operator*

$$\mathcal{T}_\psi = P_{\mathbb{T}} \psi_{|[P_{\mathbb{T}} L^p(\mathbb{T},w)]^2} : [P_{\mathbb{T}} L^p(\mathbb{T},w)]^2 \to [P_{\mathbb{T}} L^p(\mathbb{T},w)]^2 ,$$

where $\psi := (u_{\mathbb{T}} - v_{\mathbb{T}})^{-1}(u_{\mathbb{T}} + v_{\mathbb{T}})$, and $w(t) = |i\frac{1+t}{1-t}|^{-1/2p}|1-t|^{1-2/p}$.

11.4 Conditions for the Invertibility of Operator $\mathcal{A}$

As stated in the Introduction (and as a consequence of the operator relations presented in the previous section), in this section we will obtain an invertibility criterion for the operator $\mathcal{A}$ and the form of its inverse/lateral inverse (under the conditions which ensure such invertibility).

Let us start by recalling some basic definitions which will be used in what follows. Considering Banach spaces X and Y, an operator $T \in \mathcal{L}(X,Y)$ is said to be *left* (respectively *right*) *invertible* if there is an operator $T^{(-1)} \in \mathcal{L}(Y,X)$ such that

$$T^{(-1)}Tx = x, \quad x \in X \qquad (TT^{(-1)}y = y, \quad y \in Y).$$

The operator $T^{(-1)}$ is then called a *left* (respectively *right*) *inverse* of T. If an operator T is both left and right invertible, then all left and right inverses are equal to each other and coincide with the inverse T^{-1} of T. Recall also that an operator $T^- : Y \longrightarrow X$ is called a *generalized inverse* of a bounded linear operator $T : X \longrightarrow Y$ if $TT^-T = T$.

A representation of the form $A = A_- \Lambda A_+$ is called a (*right*) *generalized factorization* of the invertible matrix function $A \in [L^\infty(\mathbb{T})]^{2\times 2}$ in the space $[L^p(\mathbb{T},\sigma)]^2$ if $\Lambda(t) = \operatorname{diag}(t^{\aleph_1}, t^{\aleph_2})$ with certain integers $\aleph_1 \geq \aleph_2$ and if the factors A_- and A_+ satisfy the following conditions:

i) $A_- \in [L^p_-(\mathbb{T},\sigma)]^{2\times 2}$, $A_+ \in [L^q_+(\mathbb{T},\sigma^{-1})]^{2\times 2}$, $A_-^{-1} \in [L^q_-(\mathbb{T},\sigma^{-1})]^{2\times 2}$, $A_+^{-1} \in [L^p_+(\mathbb{T},\sigma)]^{2\times 2}$ ($\frac{1}{p} + \frac{1}{q} = 1$), where $L^p_+(\mathbb{T},\sigma) := P_{\mathbb{T}} L^p(T,\sigma)$ and $L^p_-(\mathbb{T},\sigma) := Q_{\mathbb{T}} L^p(T,\sigma) \oplus \mathbb{C}$ are subspaces of $L^p(\mathbb{T},\sigma)$, and $P_{\mathbb{T}} = \frac{1}{2}(I_{\mathbb{T}} + S_{\mathbb{T}})$ and $Q_{\mathbb{T}} = \frac{1}{2}(I_{\mathbb{T}} - S_{\mathbb{T}})$.
ii) The operator $A_- P_{\mathbb{T}} A_-^{-1}$ is bounded on the space $[L^p(\mathbb{T},\sigma)]^2$.

The integers $\aleph_i$, $i = 1,2$, are called (*right*) *indices* or also *partial indices* of the generalized factorization of the matrix function A. The sum $\aleph_1 + \aleph_2 =: \aleph$ is referred to as the *total index* or *sum index* of the matrix function A.

Theorem 3. *Let* $u_{\mathbb{T}}, v_{\mathbb{T}} \in [L^\infty(\mathbb{T})]^{2\times 2}$ *be the matrix functions given by* (11.13) *such that* $\det(u_{\mathbb{T}}(t) \pm v_{\mathbb{T}}(t)) \neq 0$, *and assume that* $\psi = (u_{\mathbb{T}} - v_{\mathbb{T}})^{-1}(u_{\mathbb{T}} + v_{\mathbb{T}})$ *admits a generalized factorization*

$$\psi(t) = \psi_-(t)\Lambda(t)\psi_+(t).$$

Then the operator $\mathcal{A}$ *is generalized invertible on the space* $L^p(\mathbb{T}, |t-1|^{1-2/p})$, $1 < p < \infty$. *A generalized inverse of* $\mathcal{A}$ *is given by*

$$\begin{aligned}\mathcal{A}^- \; = \; & B^{-1} M_{\mathbb{R}_+} K R_{\mathbb{R}_+} N_{\mathbb{R}_+} \mathrm{Rest}_{[L^p(\mathbb{R}_+,|x|^{-1/2p})]^2} \big(B_2(\psi_+^{-1} P_{\mathbb{T}} + \psi_- Q_{\mathbb{T}}) \\ & (\Lambda^{-1} P_{\mathbb{T}} + Q_{\mathbb{T}})\psi_-^{-1}(u_{\mathbb{T}} I_{\mathbb{T}} - v_{\mathbb{T}} I_{\mathbb{T}})^{-1} B_2^{-1}\big) N_{\mathbb{R}_+}^{-1} K^{-1} M_{\mathbb{R}_+}^{-1} B, \quad (11.17)\end{aligned}$$

where $B^{\pm 1}, M_{\mathbb{R}}^{\pm 1}, K, R_{\mathbb{R}_+}, N_{\mathbb{R}}^{\pm 1}$, *and* $B_2^{\pm 1}$ *are given in the last section.*

The operator $\mathcal{A}$ *is invertible* (*left-sided invertible, right-sided invertible*) *if and only if all indices of the matrix function* ψ *are zero* (*nonnegative, nonpositive*). *In such a case, the inverse* (*left inverse, right inverse*) *is also given by* (11.17) (*where in each case some simplifications occur in the formula*).

Proof. From Theorem 2 we have that the operator $\mathcal{A}$ is equivalent after extension to the operator $\mathcal{D}_{\mathbb{T}}$ given by (11.12). Now, we rewrite the operator $\mathcal{D}_{\mathbb{T}}$ in terms of the Riesz projections $P_{\mathbb{T}}$ and $Q_{\mathbb{T}}$, e.g.,

$$\mathcal{D}_{\mathbb{T}} = u_{\mathbb{T}} I_{\mathbb{T}} + v_{\mathbb{T}} S_{\mathbb{T}} = (u_{\mathbb{T}} I_{\mathbb{T}} + v_{\mathbb{T}} I_{\mathbb{T}}) P_{\mathbb{T}} + (u_{\mathbb{T}} I_{\mathbb{T}} - v_{\mathbb{T}} I_{\mathbb{T}}) Q_{\mathbb{T}}.$$

The invertibility conclusions for the operator $\mathcal{D}_{\mathbb{T}}$ are obtained from the well-known Simonenko's theorem; see, for instance, Theorem 4.2 in [MiPr80] for continuous matrix functions. This can be generalized for bounded measurable matrix functions as follows (see [MiPr80, Chapter V, Section 5]): Under the assumption that the matrix-valued function ψ admits a generalized factorization in the space $[L^p(\mathbb{T}, w)]^2$, say $\psi = \psi_- \Lambda \psi_+$, then the operator $\mathcal{D}_{\mathbb{T}}$ is generalized invertible on the space $[L^p(\mathbb{T}, w)]^2$, $1 < p < \infty$ and $w(t) = |i\frac{1+t}{1-t}|^{-1/2p}|1-t|^{1-2/p}$ with a generalized inverse given by

$$\mathcal{D}_{\mathbb{T}}^- = (\psi_+^{-1} P_{\mathbb{T}} + \psi_- Q_{\mathbb{T}})(\Lambda^{-1} P_{\mathbb{T}} + Q_{\mathbb{T}})\psi_-^{-1}(u_{\mathbb{T}} I_{\mathbb{T}} - v_{\mathbb{T}} I_{\mathbb{T}})^{-1}. \quad (11.18)$$

In the case of all right partial indices of the matrix function ψ being zero (nonnegative, nonpositive), then $\mathcal{D}_{\mathbb{T}}$ is invertible (left-sided invertible, right-sided invertible) and the inverse (left-sided inverse, right-sided inverse) is also given by (11.18).

Finally, we will use the explicit equivalence relation exhibited in Theorems 1 and 2 to obtain a generalized inverse (inverse, left inverse, right inverse) of the operator $\mathcal{A}$:

$$\mathcal{A}^- = \mathcal{V}\, \mathrm{Rest}_{[L^p(\mathbb{R}_+,|x|^{-1/2p})]^2} (B_2 \mathcal{D}_{\mathbb{T}}^- B_2^{-1})\, \mathcal{G}, \quad (11.19)$$

where the operators $B_2^{\pm 1}$, $\mathcal{G}$, and $\mathcal{V}$ are given in Theorem 1. Putting equality (11.18) into the equality (11.19) and writing the explicit form of $\mathcal{G}$ and $\mathcal{V}$, we obtain the conclusion.

11.5 Example

We end this work by considering a concrete example of an operator $\mathcal{A}$ in order to derive a corresponding conclusion about its invertibility.

Let $\mathcal{A} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 \widetilde{J} + b_1 S_{\mathbb{T}} \widetilde{J} : L^p(\mathbb{T}, \rho) \to L^p(\mathbb{T}, \rho)$, with $\rho(t) = |t-1|^{1-2/p}$ and with coefficients

$$\begin{aligned} a_0(t) &:= -\frac{1}{2}\sin^2\left(\frac{\pi}{2}t^2\right), & a_1(t) &:= \frac{1}{2}\sin\left(\frac{\pi}{2}\mathrm{sign}_{\mathbb{T}}(t)t^4\right), \\ b_0(t) &:= \frac{1}{2}t^4, & b_1(t) &:= \frac{1}{2}\sin\left(\frac{\pi}{2}\mathrm{sign}_{\mathbb{T}}(t)t^4\right), \end{aligned}$$

where $\mathrm{sign}_{\mathbb{T}}$ is defined by the rule

$$\mathrm{sign}_{\mathbb{T}}(t) := \begin{cases} 1, & \text{if } t \in \mathbb{T}_+ \\ -1, & \text{if } t \in \mathbb{T}_-. \end{cases}$$

From Theorem 2 we know that operator $\mathcal{A}$ is equivalent to the matrix operator $\mathcal{D}_{\mathbb{T}} = u_{\mathbb{T}} I_{\mathbb{T}} + v_{\mathbb{T}} S_{\mathbb{T}} : [L^p(\mathbb{T}, w)]^2 \to [L^p(\mathbb{T}, w)]^2$, where

$$w(t) = \left| i\frac{1+t}{1-t} \right|^{-1/2p} |1-t|^{1-2/p}$$

and $u_{\mathbb{T}}$, $v_{\mathbb{T}}$ are obtained as in (11.13) and defined on $\mathbb{T}$ as indicated in (11.14), (11.15), and (11.16), with

$$u_{\mathbb{T}_+}(t) = \frac{1}{2}\begin{pmatrix} \mu_{11}(t) & \mu_{12}(t) \\ \mu_{21}(t) & \mu_{22}(t) \end{pmatrix}.$$

In the present case, we have

$$\begin{aligned} \mu_{11}(t) &= \frac{1}{2}\Big[\mathrm{sign}_{\mathbb{T}}(t^{/2})t^2 + \sin\left(\frac{\pi}{2}\mathrm{sign}_{\mathbb{T}}(t^{1/2})t^2\right) \\ &\qquad - \mathrm{sign}_{\mathbb{T}}(-t^{1/2})t^2 - \sin\left(\frac{\pi}{2}\mathrm{sign}_{\mathbb{T}}(-t^{1/2})t^2\right)\Big], \\ \mu_{12}(t) &= 0, \\ \mu_{21}(t) &= \frac{1}{2}\Big[\mathrm{sign}_{\mathbb{T}}(t^{/2})t^2 + \sin\left(\frac{\pi}{2}\mathrm{sign}_{\mathbb{T}}(t^{1/2})t^2\right) \\ &\qquad + \mathrm{sign}_{\mathbb{T}}(-t^{1/2})t^2 + \sin\left(\frac{\pi}{2}\mathrm{sign}_{\mathbb{T}}(-t^{1/2})t^2\right)\Big], \\ \mu_{22}(t) &= \sin^2\left(\frac{\pi}{2}t\right) + t^2. \end{aligned}$$

Note that this yields in particular $\mu_{11}(\pm 1) = \frac{1}{2}[1+\sin\left(\frac{\pi}{2}\right)+1-\sin\left(-\frac{\pi}{2}\right)] = 2$, $\mu_{12}(\pm 1) = 0$, $\mu_{21}(\pm 1) = \frac{1}{2}[1+\sin\left(\frac{\pi}{2}\right) - 1 + \sin\left(-\frac{\pi}{2}\right)] = 0$, and $\mu_{22}(\pm 1) = 2$. Also, for the present case, we have

$$v_{\mathbb{T}_+}(t) = \frac{1}{2}\begin{pmatrix} \vartheta_{11}(t) & \vartheta_{12}(t) \\ \vartheta_{21}(t) & \vartheta_{22}(t) \end{pmatrix}$$

with $\vartheta_{11}(t) = \sin^2\left(\frac{\pi}{2}t\right) - t^2$, $\vartheta_{12}(t) = \frac{1}{2}[\mathrm{sign}_{\mathbb{T}}(t^{/2})t^2 - \sin\left(\frac{\pi}{2}\mathrm{sign}_{\mathbb{T}}(t^{1/2})t^2\right) + \mathrm{sign}_{\mathbb{T}}(-t^{1/2})t^2 - \sin\left(\frac{\pi}{2}\mathrm{sign}_{\mathbb{T}}(-t^{1/2})t^2\right)]$, $\vartheta_{21}(t) = 0$, $\vartheta_{22}(t) = \frac{1}{2}[\mathrm{sign}_{\mathbb{T}}(t^{/2})t^2 - \sin\left(\frac{\pi}{2}\mathrm{sign}_{\mathbb{T}}(t^{1/2})t^2\right) - \mathrm{sign}_{\mathbb{T}}(-t^{1/2})t^2 + \sin\left(\frac{\pi}{2}\mathrm{sign}_{\mathbb{T}}(-t^{1/2})t^2\right)]$, and satisfying $\vartheta_{11}(\pm 1) = 0$, $\vartheta_{12}(\pm 1) = \frac{1}{2}[1 - 1 - (\sin\left(\frac{\pi}{2}\right) + \sin\left(-\frac{\pi}{2}\right))] = 0$, $\vartheta_{21}(\pm 1) = 0$, $\vartheta_{22}(\pm 1) = \frac{1}{2}\left[1 + 1 - \sin\left(\frac{\pi}{2}\right) + \sin\left(-\frac{\pi}{2}\right)\right] = 0$.

Therefore, the matrix functions $u_{\mathbb{T}}$ and $v_{\mathbb{T}}$ are continuous on the whole $\mathbb{T}$. Considering now $\psi = (u_{\mathbb{T}} - v_{\mathbb{T}})^{-1}(u_{\mathbb{T}} + v_{\mathbb{T}})$, it follows (in this case) that

$$\psi(t) = \begin{pmatrix} -\frac{2\left(\frac{1}{2}\sin\left(\frac{\pi}{2}t^2\right)+\frac{1}{2}+\frac{1}{2}\sin^2\left(\frac{\pi}{2}t\right)-\frac{1}{2}t^2\right)}{-\sin\left(\frac{\pi}{2}t^2\right)-1+\sin^2\left(\frac{\pi}{2}t\right)-t^2} & 0 \\ 0 & \frac{2\left(\frac{1}{2}\sin^2\left(\frac{\pi}{2}t\right)-\frac{1}{2}\sin\left(\frac{\pi}{2}t^2\right)+t^2\right)}{\sin^2\left(\frac{\pi}{2}t\right)+\sin\left(\frac{\pi}{2}t^2\right)} \end{pmatrix},$$

for $t \in \mathbb{T}_+$, and $\psi(t) = I_{2\times 2}$ for $t \in \mathbb{T}_-$. First of all, to consider the eventual Fredholm property of the present operator $\mathcal{A}$, it is enough to study $\det(\psi)$. Computing such a determinant, we have

$$\det(\psi(t)) = -\frac{\left(-\sin\left(\frac{\pi}{2}t^2\right) - 1 - \sin^2\left(\frac{\pi}{2}t\right) + t^2\right)\left(\sin^2\left(\frac{\pi}{2}t\right) + 2t^2 - \sin\left(\frac{\pi}{2}t^2\right)\right)}{\left(\sin\left(\frac{\pi}{2}t^2\right) + 1 - \sin^2\left(\frac{\pi}{2}t\right) + t^2\right)\left(\sin^2\left(\frac{\pi}{2}t\right) + \sin\left(\frac{\pi}{2}t^2\right)\right)}$$

for $t \in \mathbb{T}_+$, and $\det(\psi(t)) = 1$ in the case of $t \in \mathbb{T}_-$. The range of $\det(\psi(t))$ is plotted in Figure 11.1.

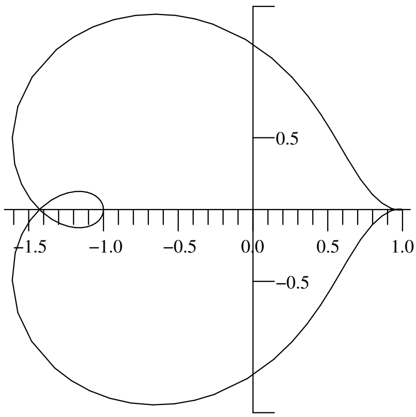

Fig. 11.1. The range of $\det(\psi)$ in the example.

Since $u_{\mathbb{T}}$ and $v_{\mathbb{T}}$ are continuous matrix-valued functions, the singular integral operator with backward Carleman shift $\mathcal{A} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 \widetilde{J} + b_1 S_{\mathbb{T}} \widetilde{J}$ acting on the space $L^p(\mathbb{T}, \rho)$, where $1 < p < \infty$ and $\rho(t) = |t - 1|^{1-2/p}$, is a Fredholm operator if and only if $\det(u_{\mathbb{T}}(t) \pm v_{\mathbb{T}}(t)) \neq 0$, $t \in \mathbb{T}$ [CaRo09].

Moreover, under the Fredholm property, the Fredholm index of $\mathcal{A}$ is given by $\operatorname{Ind}\mathcal{A} = -\text{wind}\,\det(\psi)$, where $\psi := (u_{\mathbb{T}} - v_{\mathbb{T}})^{-1}(u_{\mathbb{T}} + v_{\mathbb{T}})$ and wind $\det(\psi)$ denotes the winding number of $\det(\psi)$.

In this way, it turns out that $\mathcal{A}$ is a Fredholm operator in $L^p(\mathbb{T}, w)$ but with $\operatorname{Ind}\mathcal{A} = -1$. Thus, the present particular operator $\mathcal{A}$ is not invertible in the spaces under consideration.

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12

Contact Problems in Bending of Thermoelastic Plates

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12.1 Introduction

The theory of bending of plates with transverse shear deformation is very important in mechanical engineering because of its direct application to the study of deformable structures and because of its mathematical sophistication. The well-posedness of boundary value problems and of initial-boundary value problems with various types of boundary conditions for this model has been studied in detail in [ChCo00] and [ChCo05], respectively. Corresponding results for the same plate model where, additionally, there are significant thermal effects have been obtained in [ChEtAl04], [ChEtAl05a], [ChEtAl05b], [ChEtAl06], [ChCo07], [ChCo08a], [ChCo08b], [ChCo08c], [ChCo09a], and [ChCo09b]. Here we present the solution to the case of a piecewise homogeneous plate with transmission boundary conditions.

12.2 Formulation of the Problem

Suppose that the plate occupies a region $\bar{S} \times [-h_0/2, h_0/2]$, $\bar{S} \subset \mathbb{R}^2$. The displacement–temperature vector

$$U(x,t) = (u(x,t)^T, u_4(x,t))^T, \quad x = (x_1, x_2) \in \bar{S},$$
$$u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))^T$$

satisfies the field equations

$$\begin{aligned} \mathsf{L}U(x,t) &= \mathsf{B}_0 \partial_t^2 U(x,t) + \mathsf{B}_1 \partial_t U(x,t) + \mathsf{A}U(x,t) \\ &= \mathsf{Q}(x,t), \quad (x,t) \in S \times (0,\infty), \end{aligned}$$

where

$$\mathsf{B}_0 = \operatorname{diag}\{\rho h^2, \rho h^2, \rho, 0\}, \quad h^2 = h_0^2/12,$$

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$$\mathsf{B}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \eta\partial_1 & \eta\partial_2 & 0 & \varkappa^{-1} \end{pmatrix}, \quad \mathsf{A} = \begin{pmatrix} & & & h^2\varpi\partial_1 \\ & A & & h^2\varpi\partial_2 \\ & & & 0 \\ 0 & 0 & 0 & -\Delta \end{pmatrix},$$

$$A = \begin{pmatrix} -h^2\mu\Delta - h^2(\lambda+\mu)\partial_1^2 + \mu & -h^2(\lambda+\mu)\partial_1\partial_2 & \mu\partial_1 \\ -h^2(\lambda+\mu)\partial_1\partial_2 & -h^2\mu\Delta - h^2(\lambda+\mu)\partial_2^2 + \mu & \mu\partial_2 \\ -\mu\partial_1 & -\mu\partial_2 & -\mu\Delta \end{pmatrix},$$

ρ, ϖ, η, $\varkappa$, λ, and μ are physical constants, and Δ is the Laplacian. Without loss of generality [ChEtAl04], we consider the initial conditions

$$U(x,0) = 0, \quad \partial_t u(x,0) = 0, \quad x \in S.$$

We assume that the plate is infinite and piecewise homogeneous; that is, it is made of one material occupying an interior domain S^+ and of another one occupying an exterior domain S^-. The two domains are separated by a simple, closed, smooth curve ∂S and such that $\bar{S}^+ \cup \bar{S}^- = \mathbb{R}^2$. Both materials are homogeneous and isotropic. We write

$$\Sigma^\pm = S^\pm \times (0,\infty), \quad \Gamma = \partial S \times (0,\infty)$$

and consider the initial-boundary value problem (TC) consisting of

$$\begin{gathered} \mathsf{L}_\pm U_\pm(x,t) = \mathsf{Q}_\pm(x,t), \quad (x,t) \in \Sigma^\pm, \\ U_\pm(x,0) = 0, \quad \partial_t u_\pm(x,0) = 0, \quad x \in S^\pm, \\ U_+^+(x,t) - U_-^-(x,t) = F(x,t), \quad (x,t) \in \Gamma, \\ (\mathsf{T}_+U_+)^+(x,t) - (\mathsf{T}_-U_-)^-(x,t) = G(x,t), \quad (x,t) \in \Gamma, \end{gathered}$$

where

$$(\mathsf{T}_\pm U_\pm)(x,t) = \begin{pmatrix} (T_\pm u_\pm)(x,t) - h_\pm^2\varpi_\pm n(x) u_{\pm,4}(x,t) \\ \partial_n u_{\pm,4}(x,t) \end{pmatrix},$$

$$T = \begin{pmatrix} h_\pm^2\left[(\lambda_\pm + 2\mu_\pm)n_1\partial_1 + \mu_\pm n_2\partial_2\right] & h_\pm^2(\lambda_\pm n_1\partial_2 + \mu_\pm n_2\partial_1) & 0 \\ h_\pm^2(\mu_\pm n_1\partial_2 + \lambda_\pm n_2\partial_1) & h_\pm^2\left[(\lambda_\pm + 2\mu_\pm)n_2\partial_2 + \mu_\pm n_1\partial_1\right] & 0 \\ \mu_\pm n_1 & \mu_\pm n_2 & \mu_\pm\partial_n \end{pmatrix},$$

$n = (n_1, n_2, 0)^T$ is the unit outward normal to ∂S, ∂_n is the derivative in the direction of n, and the subscripts and superscripts $\pm$ distinguish between the constants, functions, and operators characterizing the domains S^+ and S^-.

12.3 Function Spaces

We denote the Laplace transformation by

$$(\mathcal{L}U)(x,p) = \hat{U}(x,p) = \int_0^\infty e^{-pt}U(x,t)\,dt$$

and introduce a number of function spaces that play an essential role in the subsequent analysis.

12.3.1 Spaces with a Parameter

Let $m \in \mathbb{R}$ and $p \in \mathbb{C}$. In our notation we have the following.

$H_m(\mathbb{R}^2)$: the standard Sobolev space of scalar distributions $\hat{v}_4(x)$, with norm

$$\|\hat{v}_4\|_m = \left\{ \int_{\mathbb{R}^2} (1+|\xi|^2)^m |\tilde{v}_4(\xi)|^2 d\xi \right\}^{1/2};$$

$\mathbf{H}_{m,p}(\mathbb{R}^2)$: the space of three-component vector distributions $\hat{v}(x)$, with norm

$$\|\hat{v}\|_{m,p} = \left\{ \int_{\mathbb{R}^2} (1+|\xi|^2+|p|^2)^m |\tilde{v}(\xi)|^2 d\xi \right\}^{1/2};$$

$\mathcal{H}_{m,p}(\mathbb{R}^2) = \mathbf{H}_{m,p}(\mathbb{R}^2) \times H_m(\mathbb{R}^2)$, with norm $|||\hat{V}|||_{m,p} = \|\hat{v}\|_{m,p} + \|\hat{v}_4\|_m$;

$H_m(S^\pm)$ and $\mathbf{H}_{m,p}(S^\pm)$: the spaces of the restrictions to $S^\pm$ of all $\hat{v}_4 \in H_m(\mathbb{R}^2)$ and $\hat{v} \in \mathbf{H}_{m,p}(\mathbb{R}^2)$, with norms

$$\|\hat{u}_4\|_{m;S^\pm} = \inf_{\hat{v}_4 \in H_m(\mathbb{R}^2):\hat{v}_4|_{S^\pm} = \hat{u}_4} \|\hat{v}_4\|_m,$$

$$\|\hat{u}\|_{m,p;S^\pm} = \inf_{\hat{v} \in H_{m,p}(\mathbb{R}^2):\hat{v}|_{S^\pm} = \hat{u}} \|\hat{v}\|_{m,p};$$

$\mathcal{H}_{m,p}(S^\pm) = \mathbf{H}_{m,p}(S^\pm) \times H_m(S^\pm)$, with norm

$$|||\hat{U}|||_{m,p;S^\pm} = \|\hat{u}\|_{m,p;S^\pm} + \|\hat{u}_4\|_{m;S^\pm};$$

$H_{1/2}(\partial S)$, $\mathbf{H}_{1/2,p}(\partial S)$: the spaces of the traces on ∂S of all $\hat{u}_4 \in H_1(S^\pm)$ and $\hat{u} \in \mathbf{H}_{1,p}(S^\pm)$, with norms

$$\|\hat{\varphi}_4\|_{1/2;\partial S} = \inf_{\hat{u}_4 \in H_1(S^+):\hat{u}_4|_{\partial S} = \hat{\varphi}_4} \|\hat{u}_4\|_{1;S^+},$$

$$\|\hat{\varphi}\|_{1/2,p;\partial S} = \inf_{\hat{u} \in \mathbf{H}_{1,p}(S^+):\hat{u}|_{\partial S} = \hat{\varphi}} \|\hat{u}\|_{1,p;S^\pm};$$

$\mathcal{H}_{1/2,p}(\partial S) = \mathbf{H}_{1/2,p}(\partial S) \times H_{1/2}(\partial S)$, with norm

$$|||\hat{F}|||_{1/2,p;\partial S} = \|\hat{\varphi}\|_{1/2,p;\partial S} + \|\hat{\varphi}_4\|_{1/2;\partial S};$$

$H_{-1/2}(\partial S)$, $\mathbf{H}_{-1/2,p}(\partial S)$, and $\mathcal{H}_{-1/2,p}(\partial S)$: the duals of the spaces $H_{1/2}(\partial S)$, $\mathbf{H}_{1/2,p}(\partial S)$, and $\mathcal{H}_{1/2,p}(\partial S)$ with respect to the duality generated by the inner product in $\left[L^2(\partial S)\right]^n$.

12.3.2 Transform Spaces

Let $\kappa > 0$, $k \in \mathbb{R}$, and $\mathbb{C}_\kappa = \{p = \sigma + i\tau \in \mathbb{C} : \sigma > \kappa\}$. We introduce the following spaces.

$\mathbf{H}^{\mathcal{L}}_{-1,k,\kappa}(\mathbb{R}^2)$, $\mathbf{H}^{\mathcal{L}}_{1,k,\kappa}(S^\pm)$, and $\mathbf{H}^{\mathcal{L}}_{\pm 1/2,k,\kappa}(\partial S)$: the spaces of all $\hat{q}(x,p)$, $\hat{u}(x,p)$, and $\hat{e}(x,p)$ that define holomorphic mappings

$$\hat{q} : \mathbb{C}_\kappa \to \mathbf{H}_{-1,0}(\mathbb{R}^2), \quad \hat{u} : \mathbb{C}_\kappa \to \mathbf{H}_{1,0}(S^\pm), \quad \hat{e} : \mathbb{C}_\kappa \to \mathbf{H}_{\pm 1/2,0}(\partial S),$$

with norms

$$\|\hat{q}\|^2_{-1,k,\kappa} = \sup_{\sigma>\kappa} \int_{-\infty}^{\infty} (1+|p|^2)^k \|\hat{q}(x,p)\|^2_{-1,p}\, d\tau,$$

$$\|\hat{u}\|^2_{1,k,\kappa;S^\pm} = \sup_{\sigma>\kappa} \int_{-\infty}^{\infty} (1+|p|^2)^k \|\hat{u}(x,p)\|^2_{1,p;S^\pm}\, d\tau,$$

$$\|\hat{e}\|^2_{\pm 1/2,k,\kappa;\partial S} = \sup_{\sigma>\kappa} \int_{-\infty}^{\infty} (1+|p|^2)^k \|\hat{e}(x,p)\|^2_{\pm 1/2,p;\partial S}\, d\tau;$$

$\mathcal{H}^{\mathcal{L}}_{-1,k,l,\kappa}(\mathbb{R}^2) = \mathbf{H}^{\mathcal{L}}_{-1,k,\kappa}(\mathbb{R}^2) \times H^{\mathcal{L}}_{1,l,\kappa}(\mathbb{R}^2)$, $\mathcal{H}^{\mathcal{L}}_{1,k,l,\kappa}(S^\pm) = \mathbf{H}^{\mathcal{L}}_{1,k,\kappa}(S^\pm) \times H^{\mathcal{L}}_{1,l,\kappa}(S^\pm)$, and $\mathcal{H}^{\mathcal{L}}_{\pm 1/2,k,l,\kappa}(\partial S) = \mathbf{H}^{\mathcal{L}}_{\pm 1/2,k,\kappa}(\partial S) \times H^{\mathcal{L}}_{\pm 1/2,l,\kappa}(\partial S)$: the spaces of all $\hat{V} = \{\hat{v}, \hat{v}_4\}$, $\hat{U} = \{\hat{u}, \hat{u}_4\}$, and $\hat{\mathcal{E}} = \{\hat{e}, \hat{e}_4\}$, with norms

$$|||\hat{V}|||_{-1,k,l,\kappa} = \|\hat{v}\|_{-1,k,\kappa} + \|\hat{v}_4\|_{-1,l,\kappa},$$

$$|||\hat{U}|||_{1,k,l,\kappa;S^\pm} = \|\hat{u}\|_{1,k,\kappa;S^\pm} + \|\hat{u}_4\|_{1,l,\kappa;S^\pm},$$

$$|||\hat{\mathcal{E}}|||_{\pm 1/2,k,l,\kappa;\partial S} = \|\hat{e}\|_{\pm 1/2,k,\kappa;\partial S} + \|\hat{e}_4\|_{\pm 1/2,l,\kappa;\partial S}.$$

12.3.3 Spaces of Originals

Let $\mathbb{R}^3_+ = \mathbb{R}^2 \times (0,\infty)$. We use the following notation.

$\mathcal{H}^{\mathcal{L}^{-1}}_{-1,k,l,\kappa}(\mathbb{R}^3_+)$, $\mathcal{H}^{\mathcal{L}^{-1}}_{1,k,l,\kappa}(\Sigma^\pm)$, $\mathcal{H}^{\mathcal{L}^{-1}}_{\pm 1/2,k,l,\kappa}(\Gamma)$: the spaces of the inverse Laplace transforms of the elements of $\mathcal{H}^{\mathcal{L}}_{-1,k,l,\kappa}(\mathbb{R}^2)$, $\mathcal{H}^{\mathcal{L}}_{1,k,l,\kappa}(S^\pm)$, $\mathcal{H}^{\mathcal{L}}_{\pm 1/2,k,l,\kappa}(\partial S)$, with norms

$$|||V|||_{-1,k,l,\kappa} = |||\hat{V}|||_{-1,k,l,\kappa}, \quad |||U|||_{1,k,l,\kappa;\Sigma^\pm} = |||\hat{U}|||_{1,k,l,\kappa;S^\pm},$$
$$|||\mathcal{E}|||_{\pm 1/2,k,l,\kappa;\Gamma} = |||\hat{\mathcal{E}}|||_{\pm 1/2,k,l,\kappa;\partial S}.$$

We denote by $\gamma^\pm$ the trace operators from $\Sigma^\pm$ to Γ and by $\gamma_0^\pm$ the trace operators from $\Sigma^\pm$ to $S^\pm \times \{0\}$.

12.4 The Weak Solution of (TC)

This is defined to be a pair of distributions $U(x,t) = \{U_+(x,t), U_-(x,t)\}$, where $U_\pm \in \mathcal{H}^{\mathcal{L}^{-1}}_{1,0,0,\kappa}(\Sigma^\pm)$, such that

(i) $\gamma_0^\pm u_\pm = 0$;

(ii) $\gamma^+ U_+ - \gamma^- U_- = F(x,t),\ \ (x,t) \in \Gamma$;

(iii) for all $W = \{W_+, W_-\} \in \mathrm{C}_0^\infty(\mathbb{R}^2 \times [0,\infty))$,

$$\Upsilon_+(U_+, W_+) + \Upsilon_-(U_-, W_-) = \int_0^\infty \big\{(\mathsf{Q}, W)_0 + (G, W)_{0;\partial S}\big\}\, dt,$$

where

$$\begin{aligned}\Upsilon_\pm(U_\pm, W_\pm) = \int_0^\infty & \big\{a_\pm(u_\pm, w_\pm)(\nabla u_{\pm,4}, \nabla w_{\pm,4})_{0;S^\pm} \\ & - (B_{0,\pm}^{1/2}\partial_t u_\pm, B_{0,\pm}^{1/2}\,\partial_t w_\pm)_{0;S^\pm} - \kappa_\pm^{-1}(u_{\pm,4}, \partial_t w_{\pm,4})_{0;S^\pm} \\ & - h_\pm^2 \varpi_\pm (u_{\pm,4}, \operatorname{div} w_\pm)_{0;S^\pm} - \eta_\pm(\operatorname{div} u_\pm, \partial_t w_{\pm,4})_{0;S^\pm}\big\}\, dt.\end{aligned}$$

Let $h_+^2 \varpi_+ \eta_+^{-1} = \iota_+$ and $h_-^2 \varpi_- \eta_-^{-1} = \iota_-$.

Theorem 1. *If* $\kappa > 0,\ l \ge 0$, *and*

$$\mathsf{Q} \in \mathcal{H}^{\mathcal{L}^{-1}}_{-1,l+1,l,\kappa}(\mathbb{R}^3_+), \quad F \in \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l+1,l+1,\kappa}(\Gamma), \quad G \in \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l+1,l,\kappa}(\Gamma),$$

then there is $\varepsilon > 0$ *such that for any* $\iota_\pm$ *satisfying* $|\iota_+ - \iota_-| < \varepsilon$, *problem* **(TC)** *has a unique weak solution*

$$U(x,t) = \{U_+(x,t), U_-(x,t)\}, \quad U_\pm \in \mathcal{H}^{\mathcal{L}^{-1}}_{1,l,l,\kappa}(\Sigma^\pm),$$

and

$$\begin{aligned}&|||U_+|||_{1,l,l,\kappa;\Sigma^+} + |||U_-|||_{1,l,l,\kappa;\Sigma^-} \\ &\qquad \le c\big\{|||\mathsf{Q}|||_{-1,l+1,l,\kappa} + |||F|||_{1/2,l+1,l+1,\kappa;\Gamma} + |||G|||_{-1/2,l+1,l,\kappa;\Gamma}\big\}.\end{aligned}$$

12.5 Boundary Integral Equations

Consider a matrix of fundamental solutions for the homogeneous governing system ($\mathsf{Q} = 0$); that is, a matrix $\mathcal{D}$ such that

$$\begin{gathered}\mathsf{B}_0(\partial_t^2 \mathcal{D})(x,t) + (\mathsf{B}_1 \partial_t \mathcal{D})(x,t) + (\mathsf{A}\mathcal{D})(x,t) = \delta(x,t) I, \quad (x,t) \in \mathbb{R}^3, \\ \mathcal{D}(x,t) = 0, \quad t < 0,\end{gathered}$$

where $\delta(x,t)$ is the Dirac delta and I is the identity (4×4)-matrix.

Let $\mathcal{A}$ and $\mathcal{B}$ be smooth functions with compact support on $\partial S\times\mathbb{R}$ and equal to zero for $t<0$.

We define the single-layer potential of density $\mathcal{A}$ by

$$(\mathcal{V}\mathcal{A})(x,t)=\int\limits_{\Gamma}\mathcal{D}(x-y,t-\tau)\mathcal{A}(y,\tau)\,ds_y\,d\tau.$$

$\mathcal{V}$ is extended by continuity to $\mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,k,l,\kappa}(\Gamma)$.

The double-layer potential of density $\mathcal{B}$ is defined by

$$(\mathcal{W}\mathcal{B})(x,t)=\int\limits_{\Gamma}\mathcal{P}(x,y,t-\tau)\mathcal{B}(y,\tau)\,ds_y d\tau,$$

where $\mathcal{P}(x,y,t)$ is a first-order differential operator applied to $\mathcal{D}$. $\mathcal{W}$ is extended by continuity to $\mathcal{H}^{\mathcal{L}^{-1}}_{1/2,k,l,\kappa}(\Gamma)$. We denote by $\mathcal{V}_{\pm}\mathcal{A}$ and $\mathcal{W}_{\pm}\mathcal{B}$ the potentials for the media in $\Sigma^{\pm}$.

A first representation of the solution is of the form

$$\begin{aligned}U_+(x,t)&=(\mathcal{V}_+\mathcal{A}_+)(x,t),\quad (x,t)\in\Sigma^+,\\ U_-(x,t)&=(\mathcal{V}_-\mathcal{A}_-)(x,t),\quad (x,t)\in\Sigma^-.\end{aligned}$$

This leads to a system of boundary integral equations written as

$$T_{VV}\{\mathcal{A}_+,\mathcal{A}_-\}=\{F,G\},\quad (x,t)\in\Gamma. \tag{12.1}$$

Theorem 2. *(i) For any $\kappa>0$, $l\ge 0$, and*

$$F\in\mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l+1,l+1,\kappa}(\Gamma),\quad G\in\mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l+1,l,\kappa}(\Gamma),$$

system (12.1) *has a unique solution*

$$\{\mathcal{A}_+,\mathcal{A}_-\}\in\mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l-1,l-2,\kappa}(\Gamma)\times\mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l-1,l-2,\kappa}(\Gamma),$$

which satisfies

$$|||\mathcal{A}_{\pm}|||_{-1/2,l-1,l-2,\kappa;\Gamma}\le c\big\{|||F|||_{1/2,l+1,l+1,\kappa;\Gamma}+|||G|||_{-1/2,l+1,l,\kappa;\Gamma}\big\}.$$

(ii) The representation $U=\{U_+,U_-\}$ constructed with the solution $\{\mathcal{A}_+,\mathcal{A}_-\}$ of system (12.1) *is the weak solution of problem* (TC).

The second representation of the solution is

$$\begin{aligned}U_+(x,t)&=(\mathcal{W}_+\mathcal{B}_+)(x,t),\quad (x,t)\in\Sigma^+,\\ U_-(x,t)&=(\mathcal{W}_-\mathcal{B}_-)(x,t),\quad (x,t)\in\Sigma^-.\end{aligned}$$

This leads to a system of boundary integral equations of the form

$$T_{WW}\{\mathcal{B}_+,\mathcal{B}_-\}=\{F,G\},\quad (x,t)\in\Gamma. \tag{12.2}$$

Theorem 3. *(i) For any $\kappa > 0$, $l \geq 0$, and*

$$F \in \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l+1,l+1,\kappa}(\Gamma), \quad G \in \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l+1,l,\kappa}(\Gamma),$$

system (12.2) *has a unique solution*

$$\{\mathcal{B}_+, \mathcal{B}_-\} \in \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l-1,l-1,\kappa}(\Gamma) \times \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l-1,l-1,\kappa}(\Gamma),$$

which satisfies

$$|||\mathcal{B}_\pm|||_{1/2,l-1,l-1,\kappa;\Gamma} \leq c\big\{|||F|||_{1/2,l+1,l+1,\kappa;\Gamma} + |||G|||_{-1/2,l+1,l,\kappa;\Gamma}\big\}.$$

(ii) The representation $U = \{U_+, U_-\}$ constructed with the solution $\{\mathcal{B}_+, \mathcal{B}_-\}$ of system (12.2) *is the weak solution of* (TC)*.*

The third representation of the solution is

$$U_+(x,t) = (\mathcal{V}_+\mathcal{A}_+)(x,t), \quad (x,t) \in \Sigma^+,$$
$$U_-(x,t) = (\mathcal{W}_-\mathcal{B}_-)(x,t), \quad (x,t) \in \Sigma^-.$$

The corresponding system of boundary integral equations in this case is of the form

$$T_{VW}\{\mathcal{A}_+, \mathcal{B}_-\} = \{F, G\}. \tag{12.3}$$

Theorem 4. *(i) For any $\kappa > 0$, $l \geq 0$, and*

$$F \in \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l+1,l+1,\kappa}(\Gamma), \quad G \in \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l+1,l,\kappa}(\Gamma),$$

system (12.3) *has a unique solution*

$$\{\mathcal{A}_+, \mathcal{B}_-\} \in \mathcal{H}^{\mathcal{L}^{-1}}_{-1/2,l-1,l-2,\kappa}(\Gamma) \times \mathcal{H}^{\mathcal{L}^{-1}}_{1/2,l-1,l-1,\kappa}(\Gamma),$$

which satisfies

$$\begin{aligned}|||\mathcal{A}_\pm|||_{-1/2,l-1,l-2,\kappa;\Gamma} &+ |||\mathcal{B}_\pm|||_{1/2,l-1,l-1,\kappa;\Gamma} \\ &\leq c\big\{|||F|||_{1/2,l+1,l+1,\kappa;\Gamma} + |||G|||_{-1/2,l+1,l,\kappa;\Gamma}\big\}.\end{aligned}$$

(ii) The representation $U = \{U_+, U_-\}$ constructed with the solution $\{\mathcal{B}_+, \mathcal{B}_-\}$ of system (12.3) *is the weak solution of problem* (TC)*.*

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13

On Burnett Coefficients in Periodic Media with Two Phases

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13.1 Introduction

In this chapter, we consider periodic media with a small period ε and we are interested in Burnett coefficients. These parameters are important in the study of acoustic wave propagation in such media since various physical constants associated with wave propagation (like reflection, refraction, transmission, and dispersion coefficients) are included in the Burnett coefficients.

Let us introduce some notations adopted in this work. We denote by Y the reference cell $(0, 2\pi)$, and for any real number $\gamma \in [0, 1]$, let T be any measurable subset of Y such that $|T| = \gamma|Y|$. We consider the operator

$$A \stackrel{\text{def}}{=} -\frac{d}{dy}\Big(\alpha(y)\frac{d}{dy}\Big), \quad y \in \mathbb{R},$$

where the coefficient $\alpha \in L^\infty_\#(Y)$, i.e., $\alpha = \alpha(y)$ is a Y-periodic bounded measurable function defined on $\mathbb{R}$, and in the reference cell is given by

$$\alpha(y) = \alpha_0 \chi_{T^C}(y) + \alpha_1 \chi_T(y), \quad y \in Y,$$

with $\alpha_0, \alpha_1 > 0$, $\alpha_0 \neq \alpha_1$. Here $\chi_T(y)$ denotes the characteristic function of T. For each $\varepsilon > 0$, we also consider the εY-periodic operator A^ε defined by

$$A^\varepsilon \stackrel{\text{def}}{=} -\frac{d}{dx}\Big(\alpha^\varepsilon(x)\frac{d}{dx}\Big) \quad \text{with} \quad \alpha^\varepsilon(x) \stackrel{\text{def}}{=} \alpha(\tfrac{x}{\varepsilon}), \quad x \in \mathbb{R}.$$

The homogenized and the dispersion coefficients denoted by q and d, respectively, are defined in terms of Bloch waves ψ associated with the operator A which we introduce now. Let us consider the following spectral problem parameterized by $\eta \in \mathbb{R}$: find $\lambda = \lambda(\eta) \in \mathbb{R}$ and $\psi = \psi(y;\eta) \not\equiv 0$ such that

$$\begin{cases} A\psi(\cdot;\eta) = \lambda(\eta)\psi(\cdot;\eta) \quad \text{in} \quad \mathbb{R}, \quad \psi(\cdot;\eta) \text{ is } (\eta;Y)\text{-periodic, i.e.,} \\ \psi(y + 2\pi m;\eta) = e^{2\pi i m\eta}\psi(y;\eta) \quad \forall m \in \mathbb{Z},\ y \in \mathbb{R}. \end{cases} \tag{13.1}$$

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Next, by the Floquet theory, we define $\phi(y;\eta) = e^{-iy\eta}\psi(y;\eta)$, and (13.1) can be rewritten in terms of ϕ as follows:

$$A(\eta)\phi = \lambda\phi \quad \text{in} \quad \mathbb{R}, \quad \phi \text{ is } Y\text{-periodic.} \tag{13.2}$$

Here, the operator $A(\eta)$ is called the translated operator and is defined by $A(\eta) = e^{-iy\eta}Ae^{iy\eta}$. It is well known (see [BLP78], [CPV95]) that for each $\eta \in Y' = \big[-\frac{1}{2}, \frac{1}{2}\big)$, the above spectral problem (13.2) admits a discrete sequence of eigenvalues $\lambda_m(\eta)$; their associated eigenfunctions $\phi_m(y;\eta)$ (referred to as *Bloch waves*) enable us to describe the spectral resolution of A (as an unbounded self-adjoint operator in $L^2(\mathbb{R})$) in the orthogonal basis $\{e^{iy\eta}\phi_m(y;\eta) : m \geq 1, \eta \in Y'\}$.

Let us introduce Bloch waves at the ε-scale:

$$\lambda_m^\varepsilon(\xi) = \varepsilon^{-2}\lambda_m(\eta), \quad \phi_m^\varepsilon(x;\xi) = \phi_m(y;\eta), \quad \psi_m^\varepsilon(x;\xi) = \psi_m(y;\eta),$$

where the variables (x,ξ) and (y,η) are related by $y = \frac{x}{\varepsilon}$ and $\eta = \varepsilon\xi$.

We consider a sequence $\{u^\varepsilon\}$ bounded in $H^1(\mathbb{R})$ and $f \in L^2(\mathbb{R})$ satisfying

$$A^\varepsilon u^\varepsilon = f \quad \text{in} \quad \mathbb{R}.$$

We assume that $u^\varepsilon \rightharpoonup u$ weakly in $H^1(\mathbb{R})$. The homogenization problem consists of passing to the limit, as $\varepsilon \to 0$, in the previous equation and obtaining the equation satisfied by u, namely,

$$Qu \stackrel{\text{def}}{=} -q\frac{d^2u}{dx^2} = f \quad \text{in} \quad \mathbb{R},$$

where q is a constant known as the homogenized coefficient (see [BLP78]).

A simple relation linking q with Bloch waves is the following: $q = \frac{1}{2}\lambda_1^{(2)}(0)$ (see [COV02]). At this point, it is appropriate to recall that derivatives of the first eigenvalue and eigenfunction at $\eta = 0$ exist thanks to the analyticity property established in [CV97]. To see how the dispersion coefficient d arises, let us consider the wave propagation problem in the periodic structure governed by the operator $\partial_{tt} + A^\varepsilon$. If we consider short waves of low energy with wave number satisfying $\varepsilon^2|\xi|^4 = O(1)$ and $\varepsilon^4|\xi|^6 = o(1)$, then a simplified description is obtained with the operator $\partial_{tt} + Q + \varepsilon^2 D$, where D is the fourth order operator whose symbol is $\frac{1}{4!}\lambda_1^{(4)}(0)\xi^4$ (see [COV06]). The coefficient $d = \frac{1}{4!}\lambda_1^{(4)}(0)$, which captures dispersive effects on such waves, is the dispersion coefficient and it represents a corrector to the periodic medium. It was studied in [COV06] and, in particular, the following physical space representation for it was obtained.

Proposition 1. *We have the relations*

$$\lambda_1(0) = 0, \quad \lambda_1^{(1)}(0) = 0, \quad \frac{1}{2!}\lambda_1^{(2)}(0) = q, \quad \frac{1}{3!}\lambda_1^{(3)}(0) = 0, \quad \frac{1}{4!}\lambda_1^{(4)}(0) = d,$$

where q can be explicitly expressed:

$$\frac{1}{q} = \frac{\gamma}{\alpha_1} + \frac{1-\gamma}{\alpha_0}. \tag{13.3}$$

Moreover, the dispersion coefficient d admits the following representation:

$$d = -\frac{q}{|Y|}\int_Y (X_{(T)})^2, \tag{13.4}$$

with test function $X_{(T)}$ defined by the following cell problem:

$$\begin{cases} -\dfrac{dX_{(T)}}{dy} = 1 - q\Big(\dfrac{\chi_T}{\alpha_1} + \dfrac{1-\chi_T}{\alpha_0}\Big) & \textit{in } \mathbb{R}, \\ X_{(T)} \in H^1_{\#}(Y), \quad \dfrac{1}{|Y|}\displaystyle\int_Y X_{(T)}(y)dy = 0. \end{cases} \tag{13.5}$$

The formula (13.3) shows that q does not depend on the microstructure. On the other hand, formulas (13.4)–(13.5) show explicitly how the dispersion coefficient d depends on the microstructure through the characteristic function χ_T. In order to study this dependence of the dispersion coefficient d, first, in Section 13.2 we are interested in the particular case of a low-contrast periodic structure. We expand the homogenized and dispersion coefficients with respect to the contrast parameter and we study the signs of the different coefficients in the expansions. Next, in Section 13.3 we investigate the general one-dimensional structure and we look for the optimal lower and upper bounds of the dispersion coefficient as the microstructure varies preserving the volume proportion γ. We find the set in which the dispersion coefficient lies.

13.2 Low-Contrast Periodic Structure

In this section, we assume that the periodic medium consists of a two-phase material with low contrast. More precisely, let α_0 be the constant coefficient representing the background isotropic homogeneous medium and α_1 be the corresponding coefficient for the perturbed medium. The main assumption of this section is the relation

$$\alpha_1 = (1+\delta)\alpha_0,$$

with $\delta \in \mathbb{R}$, $|\delta| << 1$ denoting the contrast parameter. We study the dependence of the homogenized and the dispersion coefficients in terms of δ. Indeed, we will expand them as a power series in δ and give explicit expressions for the coefficients of various terms of the expansion which in turn yield necessary and sufficient conditions for their signs. The significance of the coefficients is obvious: they represent the contributions of the microstructure at various orders of δ. Though, in principle, we can deal with all the coefficients appearing in the expansion, we treat only the first five of them since they are of interest to engineers (see [Tor02, page 526]).

Proposing the following expansions with respect to δ:

$$q = \sum_{k=0}^{\infty} \delta^k q^{(k)} \qquad \text{and} \qquad d = \sum_{k=0}^{\infty} \delta^k d^{(k)}, \tag{13.6}$$

we establish the following theorem.

Theorem 1. *The first five terms in the expansions* (13.6) *satisfy the following inequalities:*

$$q^{(0)} > 0, \quad q^{(1)} \geq 0, \quad q^{(2)} \leq 0, \quad q^{(3)} \geq 0, \quad q^{(4)} \leq 0, \tag{13.7}$$

$$d^{(0)} = 0, \quad d^{(1)} = 0, \quad d^{(2)} \leq 0, \tag{13.8}$$

$$d^{(3)} \geq 0 \text{ if and only if } \gamma \leq \frac{2}{3}, \quad d^{(4)} \leq 0 \text{ if and only if } \gamma \leq \frac{1}{2}. \tag{13.9}$$

Remark 1. In [CSMSV08], we have showed that the inequalities (13.7), (13.8) hold irrespective of dimensions and without any hypothesis on γ. Moreover, we have proved the inequalities (13.9) first in one dimension, and second in higher dimensions, but with coefficients varying only in one direction (under what is called the laminated microstructure hypothesis). More precisely, two examples of laminated structures referred to as *longitudinal* and *orthogonal* cases have been treated there.

Remark 2. Since the homogenized and dispersion coefficients depend on the microstructure, so do their signs. Our finding is that this dependence is only through the local proportion parameter γ. It is worth remarking that this parameter plays a crucial role in various optimal design problems involving microstructures (see [Mil02], [MT97]). When $\gamma = 0$ it is easy to see that d and hence, $d^{(3)}$ and $d^{(4)}$ vanish. It is a surprise to observe that as soon as γ is positive and small, the coefficients $d^{(3)}$ and $d^{(4)}$ pick up opposite signs. Results analogous to (13.9) in higher dimensions are open.

Proof. Using the representation (13.3) and the hypothesis $\alpha_1 = (1+\delta)\alpha_0$, it is straightforward to get the expansion for q. More precisely, we obtain

$$q^{(0)} = \alpha_0 > 0, \quad q^{(1)} = \alpha_0 \gamma \geq 0, \quad q^{(2)} = -\alpha_0 \gamma (1-\gamma) \leq 0, \tag{13.10}$$

$$q^{(3)} = \alpha_0 \gamma (1-\gamma)^2 \geq 0, \quad q^{(4)} = -\alpha_0 \gamma (1-\gamma)^3 \leq 0. \tag{13.11}$$

This concludes the proof of the inequalities (13.7).

On the other hand, proposing the ansatz

$$X_{(T)} = X_{(T)}^{(0)} + \delta X_{(T)}^{(1)} + \delta^2 X_{(T)}^{(2)} + \delta^3 X_{(T)}^{(3)} + \delta^4 X_{(T)}^{(4)} + \cdots \tag{13.12}$$

and using the expansion of q in the representation formula (13.4) of the dispersion coefficient, we have

$$d = -\frac{1}{|Y|}\sum_{i,j,k=0}^{\infty} \delta^{i+j+k} q^{(i)} \int_Y X_{(T)}^{(j)} X_{(T)}^{(k)}.$$

Therefore, the coefficients $d^{(j)}$, $j \in \{0,1,2,3,4\}$ in (13.6) are given by

$$d^{(0)} = -\frac{q^{(0)}}{|Y|}\int_Y \left(X_{(T)}^{(0)}\right)^2, \tag{13.13}$$

$$d^{(1)} = -\frac{1}{|Y|}\left[2q^{(0)}\int_Y X_{(T)}^{(0)}X_{(T)}^{(1)} + q^{(1)}\int_Y \left(X_{(T)}^{(0)}\right)^2\right], \tag{13.14}$$

$$\begin{aligned} d^{(2)} = -\frac{1}{|Y|}\Bigg[&2q^{(0)}\int_Y X_{(T)}^{(0)}X_{(T)}^{(2)} + q^{(0)}\int_Y \left(X_{(T)}^{(1)}\right)^2 \\ &+ 2q^{(1)}\int_Y X_{(T)}^{(0)}X_{(T)}^{(1)} + q^{(2)}\int_Y \left(X_{(T)}^{(0)}\right)^2\Bigg], \end{aligned} \tag{13.15}$$

$$\begin{aligned} d^{(3)} = -\frac{1}{|Y|}\Bigg[&2q^{(0)}\int_Y X_{(T)}^{(0)}X_{(T)}^{(3)} + 2q^{(0)}\int_Y X_{(T)}^{(1)}X_{(T)}^{(2)} + 2q^{(1)}\int_Y X_{(T)}^{(0)}X_{(T)}^{(2)} \\ &+ q^{(1)}\int_Y \left(X_{(T)}^{(1)}\right)^2 + 2q^{(2)}\int_Y X_{(T)}^{(0)}X_{(T)}^{(1)} \\ &+ q^{(3)}\int_Y \left(X_{(T)}^{(0)}\right)^2\Bigg], \end{aligned} \tag{13.16}$$

$$\begin{aligned} d^{(4)} = -\frac{1}{|Y|}\Bigg[&2q^{(0)}\int_Y X_{(T)}^{(0)}X_{(T)}^{(4)} + 2q^{(0)}\int_Y X_{(T)}^{(1)}X_{(T)}^{(3)} + q^{(0)}\int_Y \left(X_{(T)}^{(2)}\right)^2 \\ &+ 2q^{(1)}\int_Y X_{(T)}^{(0)}X_{(T)}^{(3)} + 2q^{(1)}\int_Y X_{(T)}^{(1)}X_{(T)}^{(2)} + 2q^{(2)}\int_Y X_{(T)}^{(0)}X_{(T)}^{(2)} \\ &+ q^{(2)}\int_Y \left(X_{(T)}^{(1)}\right)^2 + 2q^{(3)}\int_Y X_{(T)}^{(0)}X_{(T)}^{(1)} + q^{(4)}\int_Y \left(X_{(T)}^{(0)}\right)^2\Bigg]. \end{aligned} \tag{13.17}$$

Let us now establish some crucial relations. Recalling that $X_{(T)}$ satisfies equation (13.5), using the ansatz (13.12), and identifying the various powers of δ, we have the following results.

Lemma 1. *The following identities hold:*

$$X_{(T)}^{(0)} \equiv 0, \tag{13.18}$$

$$-\frac{dX_{(T)}^{(1)}}{dy} = \chi_T - \gamma \quad in\ Y, \tag{13.19}$$

$$X_{(T)}^{(j+1)} = -(1-\gamma)X_{(T)}^{(j)} \qquad \forall j \in \mathbb{N}^*. \tag{13.20}$$

As direct consequences of this lemma we get the following corollary.

Corollary 1. *The following relations are true:*

$$X_{(T)}^{(j)} = (-1)^{j-1}(1-\gamma)^{j-1}X_{(T)}^{(1)} \qquad \forall j \in \mathbb{N}^*,$$

$$\int_Y X_{(T)}^{(j)}X_{(T)}^{(k)} = (-1)^{j+k}(1-\gamma)^{j+k-2}\int_Y \left(X_{(T)}^{(1)}\right)^2 \qquad \forall j,k \in \mathbb{N}^*.$$

Using these results, let us prove the inequalities (13.8) and (13.9). On one hand, since (13.13)–(13.15) and (13.18) hold, we easily deduce $d^{(0)} = d^{(1)} = 0$ and

$$d^{(2)} = -\frac{q^{(0)}}{|Y|}\int_Y \left(X^{(1)}_{(T)}\right)^2 \leq 0.$$

Thus, we prove relations (13.8).

On the other hand, again using (13.18), formulas (13.16) and (13.17) become

$$\begin{aligned} d^{(3)} &= -\frac{1}{|Y|}\Big[2q^{(0)}\int_Y X^{(1)}_{(T)}X^{(2)}_{(T)} + q^{(1)}\int_Y \left(X^{(1)}_{(T)}\right)^2\Big], \\ d^{(4)} &= -\frac{1}{|Y|}\Big[2q^{(0)}\int_Y X^{(1)}_{(T)}X^{(3)}_{(T)} + q^{(0)}\int_Y \left(X^{(2)}_{(T)}\right)^2 \\ &\qquad +2q^{(1)}\int_Y X^{(1)}_{(T)}X^{(2)}_{(T)} + q^{(2)}\int_Y \left(X^{(1)}_{(T)}\right)^2\Big]. \end{aligned}$$

Due to Corollary 1 and the expressions of $q^{(j)}$ given in (13.10)–(13.11), we get

$$d^{(3)} = \alpha_0(2-3\gamma)\frac{1}{|Y|}\int_Y \left(X^{(1)}_{(T)}\right)^2,\ d^{(4)} = -3\alpha_0(1-\gamma)(1-2\gamma)\frac{1}{|Y|}\int_Y \left(X^{(1)}_{(T)}\right)^2.$$

Then, it follows easily that $d^{(3)} \geq 0$ if and only if $\gamma \leq \frac{2}{3}$ and $d^{(4)} \leq 0$ if and only if $\gamma \leq \frac{1}{2}$, and we conclude the proof of inequalities (13.9).

Remark 3. The expressions of the coefficients $d^{(i)}$, $i \in \{2, 3, 4\}$ depend on the microstructure through the integral $\int_Y (X^{(1)}_{(T)})^2$.

One can give explicit formulas for these coefficients in some particular cases. For instance, for a given $n \in \mathbb{N}^*$, if we consider a multilayered mixture of the two phases, that is, $T = \bigcup_{k=0}^{n-1} \left[\frac{k}{n}|Y|, \frac{k+\gamma}{n}|Y|\right]$, then $\int_Y (X^{(1)}_{(T)})^2 = \frac{|Y|^3}{12n^2}\gamma^2(1-\gamma)^2$. Therefore,

$$d^{(2)} = -\tfrac{\alpha_0}{12n^2}|Y|^2\gamma^2(1-\gamma)^2, \qquad d^{(3)} = \tfrac{\alpha_0}{12n^2}|Y|^2\gamma^2(1-\gamma)^2(2-3\gamma),$$

$$d^{(4)} = -\tfrac{\alpha_0}{4n^2}|Y|^2\gamma^2(1-\gamma)^3(1-2\gamma).$$

13.3 Optimal Bounds for the Burnett Coefficient

In this section, we assume that the periodic medium with two phases is a general one. The purpose of this section is to find the set in which the dispersion coefficient d lies, as the microstructure varies, preserving the volume proportion γ. Let us first observe that if $\gamma \in \{0, 1\}$, the dispersion coefficient d is equal to 0. For this reason, we take $\gamma \in (0, 1)$ in the sequel.

Let us introduce some notations. We denote by $\mathrm{Char}(Y)$ the set of all characteristic functions of measurable subsets of Y, and for any $\boldsymbol{\chi} \in \mathrm{Char}(Y)$,

$T(\chi) = \{y \in Y : \chi(y) = 1\}$. For a given $\gamma \in (0,1)$, the set $\mathcal{C}_\gamma$ of classical microstructures is given by

$$\mathcal{C}_\gamma \quad = \quad \left\{\chi \in \mathrm{Char}(Y) \;:\; |T(\chi)| = \gamma |Y|\right\}.$$

In Char(Y), we define the functional J_0 as follows:

$$J_0 \colon \mathrm{Char}(Y) \longrightarrow \mathbb{R}, \quad J_0(\chi) \stackrel{\mathrm{def}}{=} \boldsymbol{m}\big((X_{(T(\chi))})^2\big),$$

where $\boldsymbol{m}(f)$ denotes the average of f over Y and $X_{(T(\chi))}$ is the solution of equation (13.5). Using this notation, the dispersion coefficient given in (13.4) can be rewritten as $d(\chi_T) = -qJ_0(\chi_T)$; therefore, it is obvious that

$$-q \sup_{\chi \in \mathcal{C}_\gamma} J_0(\chi) \leq d(\chi_T) \leq -q \inf_{\chi \in \mathcal{C}_\gamma} J_0(\chi) \qquad \forall \chi_T \in \mathcal{C}_\gamma. \tag{13.21}$$

When dealing with minimization and maximization problems involving microstructures, of the form $\inf_{\chi \in \mathcal{C}_\gamma} J_0(\chi)$ and $\sup_{\chi \in \mathcal{C}_\gamma} J_0(\chi)$, it is known that they do not, in general, admit solutions within the class of classical microstructures. To overcome this, the proposed way is relaxation, which amounts to passage from classical to generalized microstructures. The relaxation process in our problem has been proved in [CSMSV], and we have obtained that

$$\inf_{\chi \in \mathcal{C}_\gamma} J_0(\chi) = \min_{\theta \in \mathcal{D}_\gamma} J(\theta), \qquad \sup_{\chi \in \mathcal{C}_\gamma} J_0(\chi) = \max_{\theta \in \mathcal{D}_\gamma} J(\theta). \tag{13.22}$$

Here, $\mathcal{D}_\gamma$ represents the set of generalized microstructures defined by

$$\mathcal{D}_\gamma \quad = \quad \left\{\theta \in L^\infty_\#(Y;[0,1]) \;:\; \boldsymbol{m}(\theta) = \gamma\right\},$$

and the functional J is the extension of J_0 over $L^\infty_\#(Y;[0,1])$ defined as follows:

$$J \colon L^\infty_\#(Y;[0,1]) \longrightarrow \mathbb{R}, \quad J(\theta) \stackrel{\mathrm{def}}{=} \boldsymbol{m}\big((X_\theta)^2\big),$$

where X_θ is the solution of the following relaxed version of the problem (13.5):

$$\begin{cases} -\dfrac{dX_\theta}{dy} = 1 - q(\boldsymbol{m}(\theta))\Big(\dfrac{\theta}{\alpha_1} + \dfrac{1-\theta}{\alpha_0}\Big) & \text{in } \mathbb{R}, \\ X_\theta \in H^1_\#(Y), \; \boldsymbol{m}(X_\theta) = 0, \end{cases} \tag{13.23}$$

and $q(\cdot)$ is defined by $\frac{1}{q(\tau)} = \frac{\tau}{\alpha_1} + \frac{1-\tau}{\alpha_0}$.

Let us now state the main result of this section. We compute optimal lower and upper bounds on the dispersion coefficient $d(\chi)$ for all microstructures $\chi \in \mathcal{C}_\gamma$. Moreover, we go further and we prove that the dispersion coefficient fills up an interval.

Theorem 2. *For any $\gamma \in (0,1)$, the following equality holds:*

$$\Big\{d(\chi) : \chi \in \mathcal{C}_\gamma\Big\} = \Big[-\frac{1}{12} q^3 \gamma^2 (1-\gamma)^2 |Y|^2 \Big(\frac{1}{\alpha_1} - \frac{1}{\alpha_0}\Big)^2, 0\Big).$$

In the remainder of the chapter, we give the main steps of the proof of Theorem 2. For more details, we refer the reader to [CSMSV].

Step 1: Minimization of J on $\mathcal{D}_\gamma$. Using the definition of J, it is clear that $J(\theta) \geq 0$ for all $\theta \in \mathcal{D}_\gamma$. Moreover, there exists $\theta^*_{min} \in \mathcal{D}_\gamma$ such that $J(\theta^*_{min}) = 0$, i.e., $X_{\theta^*_{min}} = 0$. More precisely, using (13.23), we get $\theta^*_{min}(y) = \gamma$. Thus, we obtain

$$\min_{\theta \in \mathcal{D}_\gamma} J(\theta) = 0. \tag{13.24}$$

Step 2: Maximization of J on $\mathcal{D}_\gamma$. First of all, since $\mathcal{D}_\gamma$ is compact with respect to the weak* topology on $L^\infty(Y)$ and J is continuous, maximizers for J over $\mathcal{D}_\gamma$ do exist. To get information on them, since in our problem we have the constraint $\boldsymbol{m}(\theta) = \gamma$, we use a Lagrange multiplier λ and introduce a Lagrangian $L(\theta, \lambda)$ as follows:

$$L(\theta, \lambda) = J(\theta) + \lambda\big(\boldsymbol{m}(\theta) - \gamma\big) \qquad \forall \theta \in L^\infty_\#(Y; [0,1]),\ \forall \lambda \in \mathbb{R}. \tag{13.25}$$

Generally, the optimality condition at a maximizer is expressed in terms of the derivative of L. As a first step, we proceed to compute the derivative via the introduction of the adjoint state equation: for all $\theta \in L^\infty_\#(Y; [0,1])$, let Q_θ be the solution of the problem

$$\begin{cases} -\dfrac{dQ_\theta}{dy} = 2q\left(\dfrac{1}{\alpha_1} - \dfrac{1}{\alpha_0}\right) X_\theta \quad \text{in } \mathbb{R}, \\ Q_\theta \in H^1_\#(Y),\ \boldsymbol{m}(Q_\theta) = 0. \end{cases} \tag{13.26}$$

For a given $\theta^* \in \mathcal{D}_\gamma$, we use this adjoint state equation with $\theta = \theta^*$ and we get that, for all $\theta \in L^\infty_\#(Y; [0,1])$ and $\lambda \in \mathbb{R}$,

$$\begin{aligned} D_\theta L(\theta^*, \lambda)(\theta - \theta^*) = {} & \boldsymbol{m}\Big(Q_{\theta^*}(\theta - \theta^*)\Big) \\ & + \Big[\lambda - q\big(\tfrac{1}{\alpha_1} - \tfrac{1}{\alpha_0}\big)\boldsymbol{m}(Q_{\theta^*}\theta^*)\Big]\boldsymbol{m}(\theta - \theta^*). \end{aligned} \tag{13.27}$$

In [CSMSV], we have proved that for each $\theta^* \in \mathcal{D}_\gamma$ with $J(\theta^*) = \max_{\theta \in \mathcal{D}_\gamma} J(\theta)$, there exists $\lambda^* \in \mathbb{R}$ such that

$$D_\theta L(\theta^*, \lambda^*)(\theta - \theta^*) \quad \leq \quad 0 \qquad \forall \theta \in L^\infty_\#(Y; [0,1]). \tag{13.28}$$

Using this property, we now state the following optimality condition.

Proposition 2. *For each $\theta^* \in \mathcal{D}_\gamma$ with $J(\theta^*) = \max_{\theta \in \mathcal{D}_\gamma} J(\theta)$, there exists $p^* \in \mathbb{R}$ such that the following optimality condition holds:*

$$\begin{cases} \theta^* \in [0,1] & a.e. \text{ in } \mathcal{A}(\theta^*, p^*), \\ \theta^* = 1 & a.e. \text{ in } \mathcal{B}(\theta^*, p^*), \\ \theta^* = 0 & a.e. \text{ in } \mathcal{C}(\theta^*, p^*), \end{cases} \tag{13.29}$$

where the sets $\mathcal{A}(\theta^*, p^*)$, $\mathcal{B}(\theta^*, p^*)$, *and* $\mathcal{C}(\theta^*, p^*)$ *are defined by*

$$\mathcal{A}(\theta^*, p^*) \quad = \quad \{y \in \mathbb{R} : Q_{\theta^*}(y) = p^*\}, \tag{13.30}$$

$$\mathcal{B}(\theta^*, p^*) \quad = \quad \{y \in \mathbb{R} : Q_{\theta^*}(y) > p^*\}, \tag{13.31}$$

$$\mathcal{C}(\theta^*, p^*) \quad = \quad \{y \in \mathbb{R} : Q_{\theta^*}(y) < p^*\}. \tag{13.32}$$

Proof. Combining (13.27) and (13.28), we have

$$\int_Y \Big(Q_{\theta^*}(y) - p^*\Big)(\theta(y) - \theta^*(y))dy \quad \leq \quad 0 \qquad \forall \theta \in L^\infty_\#(Y; [0,1]), \tag{13.33}$$

where $p^* = -\lambda^* + q\big(\frac{1}{\alpha_1} - \frac{1}{\alpha_0}\big)\boldsymbol{m}(Q_{\theta^*}\theta^*)$. From the integral inequality (13.33), we now deduce some pointwise information on θ^*. In the sequel, we prove that $\theta^* = 1$ almost everywhere in $\mathcal{B}(\theta^*, p^*) \cap Y$. To this end, we define the set $E = \Big\{y \in \mathcal{B}(\theta^*, p^*) \cap Y \ : \ \theta^*(y) < 1\Big\}$ and the function $\theta_E = \theta^* + (1 - \theta^*)\boldsymbol{\chi}_E$. Using this test function in inequality (13.33), we obtain

$$\int_E \big(Q_{\theta^*}(y) - p^*\big)(1 - \theta^*(y))dy \leq 0.$$

Since $\big(Q_{\theta^*}(y) - p^*\big)(1 - \theta^*(y)) > 0$ for all $y \in E$, we deduce that E is a null set and so $\theta^* = 1$ almost everywhere in $\mathcal{B}(\theta^*, p^*) \cap Y$.

Analogously, one can prove $\theta^* = 0$ almost everywhere in $\mathcal{C}(\theta^*, p^*) \cap Y$. Hence, by periodicity we get (13.29), and so the proposition is proved.

Using Proposition 2, we are now able to deduce a new expression of J evaluated in those points $\theta^* \in \mathcal{D}_\gamma$ where the optimality condition (13.29) holds. To this end, we define the set

$$\Theta_\gamma = \Big\{\theta^* \in \mathcal{D}_\gamma : \text{ there exists } p^* \in \mathbb{R} \text{ such that (13.29) holds}\Big\}. \tag{13.34}$$

For any $(\theta^*, p^*) \in \Theta_\gamma \times \mathbb{R}$ such that (13.29) holds, the following properties hold (for details, see [CSMSV]): for a given $y_{\mathcal{A}} \in \mathcal{A}(\theta^*, p^*)$ there exist two collections of disjoint open intervals $\big\{(a_i, b_i)\big\}_{i=1}^{N_{\mathcal{B}}}$ and $\big\{(c_j, d_j)\big\}_{j=1}^{N_{\mathcal{C}}}$ such that

$$\mathcal{B}(\theta^*, p^*) \cap (y_{\mathcal{A}} + Y) = \bigcup_{i=1}^{N_{\mathcal{B}}} (a_i, b_i), \quad \mathcal{C}(\theta^*, p^*) \cap (y_{\mathcal{A}} + Y) = \bigcup_{j=1}^{N_{\mathcal{C}}} (c_j, d_j), \tag{13.35}$$

where $N_{\mathcal{B}}, N_{\mathcal{C}} \in \mathbb{N} \cup \{+\infty\}$ and $a_i, b_i, c_j, d_j \in \mathcal{A}(\theta^*, p^*)$ for all $i \in \{1, \ldots, N_{\mathcal{B}}\}$, $j \in \{1, \ldots, N_{\mathcal{C}}\}$. Moreover, we have

$$\sum_{i=1}^{N_{\mathcal{B}}} (b_i - a_i) \leq \gamma |Y| \quad \text{and} \quad \sum_{j=1}^{N_{\mathcal{C}}} (d_j - c_j) \leq (1 - \gamma)|Y|. \tag{13.36}$$

Thanks to this decomposition, we can give the new expression for J on the set Θ_γ (for the proof of the following proposition, we refer the reader to [CSMSV]).

Proposition 3. *For any* $(\theta^*, p^*) \in \Theta_\gamma \times \mathbb{R}$ *such that* (13.29) *holds and* $y_{\mathcal{A}} \in \mathcal{A}(\theta^*, p^*)$*, we have*

$$J(\theta^*) = \frac{q^2}{12|Y|}\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0}\right)^2 \left[(1-\gamma)^2 \sum_{i=1}^{N_{\mathcal{B}}} (b_i - a_i)^3 + \gamma^2 \sum_{j=1}^{N_{\mathcal{C}}} (d_j - c_j)^3\right]. \quad (13.37)$$

In particular, the above expression is valid at maximizers θ^**.*

We now use the new expression of J given in Proposition 3 in order to deduce that for all $\theta^* \in \Theta_\gamma$, $J(\theta^*)$ is equal to

$$\frac{q^2}{12}\gamma^2(1-\gamma)^2|Y|^2\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0}\right)^2\left[\gamma \sum_{i=1}^{N_{\mathcal{B}}}\left(\frac{b_i - a_i}{\gamma|Y|}\right)^3 + (1-\gamma)\sum_{j=1}^{N_{\mathcal{C}}}\left(\frac{d_j - c_j}{(1-\gamma)|Y|}\right)^3\right].$$

Then, due to inequalities (13.36), we deduce the following bound for all $\theta^* \in \Theta_\gamma$:

$$J(\theta^*) \le \frac{q^2}{12}\gamma^2(1-\gamma)^2|Y|^2\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0}\right)^2\left[\gamma \sum_{i=1}^{N_{\mathcal{B}}}\frac{b_i - a_i}{\gamma|Y|} + (1-\gamma)\sum_{j=1}^{N_{\mathcal{C}}}\frac{d_j - c_j}{(1-\gamma)|Y|}\right],$$

which implies

$$J(\theta^*) \quad \le \quad \frac{q^2}{12}\gamma^2(1-\gamma)^2|Y|^2\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0}\right)^2 \qquad \forall \theta^* \in \Theta_\gamma. \quad (13.38)$$

Considering the function $\theta^*_{max} = \chi_{[0,\gamma|Y|]} \in \Theta_\gamma$, it is easy to see that

$$J(\theta^*_{max}) \quad = \quad \frac{q^2}{12}\gamma^2(1-\gamma)^2|Y|^2\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0}\right)^2. \quad (13.39)$$

Finally, we combine (13.38) with (13.39) and we obtain $\max_{\theta^* \in \Theta_\gamma} J(\theta^*) = \frac{1}{12}q^2\gamma^2(1-\gamma)^2|Y|^2\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0}\right)^2$.

As a consequence of optimality condition (13.29), we have that all maximizers of J over $\mathcal{D}_\gamma$ lie in Θ_γ and so $\max_{\theta \in \mathcal{D}_\gamma} J(\theta) = \max_{\theta^* \in \Theta_\gamma} J(\theta^*)$. Thus, we get

$$\max_{\theta \in \mathcal{D}_\gamma} J(\theta) = \frac{1}{12}q^2\gamma^2(1-\gamma)^2|Y|^2\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0}\right)^2. \quad (13.40)$$

It is surprising to find a classical microstructure θ^*_{max} as a maximizer. It follows that $J(\theta^*_{max}) = J_0(\theta^*_{max}) = \frac{1}{12}q^2\gamma^2(1-\gamma)^2|Y|^2\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0}\right)^2$.

Thus, using (13.22), (13.24), and (13.40) in (13.21), we conclude that

$$\left\{d(\chi) : \chi \in \mathcal{C}_\gamma\right\} \subseteq \left[-\frac{1}{12}q^3\gamma^2(1-\gamma)^2|Y|^2\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0}\right)^2, 0\right).$$

Step 3: Reverse inclusion. We refer the reader to our recent paper [CSMSV] for the proof of the fact that for any real number $D_0 \in \big[-\frac{1}{12}q^3\gamma^2(1-\gamma)^2|Y|^2\big(\frac{1}{\alpha_1}-\frac{1}{\alpha_0}\big)^2, 0\big)$, there exists a composite material defined by a characteristic function $\chi \in \mathcal{C}_\gamma$ such that $d(\chi) = D_0$. That is,

$$\Big\{d(\chi) : \chi \in \mathcal{C}_\gamma\Big\} \supseteq \Big[-\frac{1}{12}q^3\gamma^2(1-\gamma)^2|Y|^2\Big(\frac{1}{\alpha_1}-\frac{1}{\alpha_0}\Big)^2, 0\Big).$$

In consequence, we get that the dispersion coefficient fills up the above interval and we conclude the proof of Theorem 2.

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14

On Regular and Singular Perturbations of the Eigenelements of the Laplacian

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14.1 Introduction

We discuss the concept of singularly perturbed eigenvalue problems for the Laplace operator.

Typical problems are the boundary value problems for the eigenvalue equations in bounded domains and the possible perturbations are a small parameter at higher derivatives, small holes, thin slits, thin appendices, frequent alternation of boundary conditions, etc. The main feature of these problems is that there exists no change of variables reducing them to problems in a fixed domain with a regularly perturbed operator. At the same time, the eigenvalues of such singularly perturbed boundary value problems converge to those of certain limiting problems. This is why, in the sense of convergence, the eigenvalues behave in the regular way.

On the other hand, it is known that a regular perturbation of the waveguides can generate new eigenvalues. The opposite situation occurs for the Helmholtz resonators and their analogues, when the perturbed problem has no eigenvalues, while the limiting one has. So, in both cases the eigenvalues behave "nonregularly" in the sense of the convergence.

In what follows a problem that is singular in the first sense but regular in the second one will be called singular-regular. And vice versa, if a problem is singular in the second sense but regular in the first one, it will be called regular-singular. If the problem is regular in both senses, it will be called twice-regular. And in the same way, a problem singular in both senses will be called twice-singular.

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14.2 The Twice-Regular Case: The Schrödinger Operator with a Large Potential Concentrated on a Small Set

Let Ω be a connected bounded domain in $\mathbb{R}^n$ containing the origin and having infinitely differentiable boundary Γ, $n \geqslant 2$, λ_0 is a simple eigenvalue of a boundary value problem

$$-\Delta\psi_0 = \lambda_0\psi_0 \quad \text{in} \quad \Omega, \qquad \frac{\partial\psi_0}{\partial\mathbf{n}} = 0 \quad \text{on} \quad \Gamma, \tag{14.1}$$

where $\mathbf{n}$ is the normal, and ψ_0 is the associated eigenfunction normalized in $L^2(\Omega)$.

Consider a perturbed boundary value problem

$$\left(-\Delta + \varepsilon^{-\alpha}V\left(\frac{x}{\varepsilon}\right)\right)\psi^\varepsilon = \lambda^\varepsilon\psi^\varepsilon \quad \text{in} \quad \Omega, \quad \psi^\varepsilon = 0 \quad \text{on} \quad \Gamma, \tag{14.2}$$

where $V \in C_0^\infty(\Omega)$, $0 < \varepsilon \ll 1$.

The aim of this section is to describe the short scheme of the proofs of the following statement on the base of the regular perturbation theory.

Theorem 1. *Let $\alpha < 1$ be an arbitrary fixed number. The asymptotics of the eigenvalue λ^ε of the boundary value problem (14.2) converging to λ_0 as $\varepsilon \to 0$ is as follows:*

$$\lambda^\varepsilon = \lambda_0 + \varepsilon^{n-\alpha}\left(\psi_0^2(0)\langle V\rangle + o(1)\right), \tag{14.3}$$

where

$$\langle F\rangle = \int\limits_{\mathbb{R}^n} F(x)\mathrm{d}x.$$

We denote by $\mathcal{B}(L^2(\Omega), L^2(\Omega))$ (by $\mathcal{B}(L^2(\Omega), H_2(\Omega))$ the Banach space of linear operators from $L^2(\Omega)$ to $L^2(\Omega)$ (to $H_2(\Omega)$), and by $\mathcal{B}^{hol}(L^2(\Omega))$ we indicate the set of holomorphic operator-valued functions with values in $\mathcal{B}(L^2(\Omega), L^2(\Omega))$.

We will employ the symbol $\mathcal{A}(\lambda)$ to indicate the linear operator mapping a function $g \in L^2(\Omega)$ into solution u_0 of the boundary value problem,

$$-\Delta u_0 = \lambda u_0 - g \quad \text{in} \quad \Omega, \qquad \frac{\partial u_0}{\partial\mathbf{n}} = 0 \quad \text{on} \quad \Gamma. \tag{14.4}$$

It is known (cf., for instance, [Ka66]), that for λ close to λ_0 the operator $\mathcal{A}(\lambda)$ can be represented as

$$\mathcal{A}(\lambda) = \frac{(\bullet, \psi_0)}{\lambda - \lambda_0}\psi_0 + \widetilde{\mathcal{A}}(\lambda),$$

where

$$\widetilde{\mathcal{A}}(\lambda) \in \mathcal{B}^{hol}(L^2(\Omega)), \tag{14.5}$$

and $(\bullet, \bullet)$ indicates the scalar product in $L^2(\Omega)$. Moreover, it follows from the smoothness-improving theorems for the solutions of the problem (14.4) that

$$\widetilde{\mathcal{A}}(\lambda) \in \mathcal{B}(L^2(\Omega), H_2(\Omega)). \tag{14.6}$$

We denote by $\|\bullet\|$ and $\|\bullet\|_1$ the norms in $L^2(\Omega)$ and $H_1(\Omega)$, respectively.

Let G be a bounded domain such that $\overline{G} \subset \Omega$, and let εG be a contraction of the set G in ε^{-1} times. Employing the analogues of Hardy inequalities in [OlIo92] and [OlSa91], it is easy to show that for any function $u \in H_1(\Omega)$ the inequality

$$\int\limits_{\varepsilon G} |u|^2 \mathrm{d}x \leqslant C_1 \beta_n(\varepsilon) \|u\|_1^2 \tag{14.7}$$

holds true, where $\beta_2(\varepsilon) = \varepsilon^2 |\ln \varepsilon|$, $\beta_n(\varepsilon) = \varepsilon^2$ as $n \geqslant 3$, and the constant C_1 is independent of ε. By simple calculations one can derive from (14.6), (14.7) the estimate

$$\int\limits_{\varepsilon G} \left| \widetilde{\mathcal{A}}(\lambda) g(x) \right|^2 \mathrm{d}x \leqslant C_2 \beta_n(\varepsilon) \|g\|^2. \tag{14.8}$$

We denote by V_ε the operator of multiplication by $V(\varepsilon^{-1}x)$ and let

$$T_\varepsilon(\lambda) := \varepsilon^{-\alpha} V_\varepsilon \widetilde{\mathcal{A}}(\lambda).$$

The definition of $T_\varepsilon(\lambda)$, (14.5), and (14.8) yield

$$T_\varepsilon(\lambda) \in \mathcal{B}^{hol}(L_2(\Omega)), \qquad \|T_\varepsilon(\lambda)\|^2 \leqslant C_3 \varepsilon^{-2\alpha} \beta_n(\varepsilon). \tag{14.9}$$

We indicate

$$S_\varepsilon(\lambda) := (I - T_\varepsilon(\lambda))^{-1},$$

where I is the identity mapping. Using (14.9), it is easy to check the following.

Lemma 1. *Let $\alpha < 1$. Then for λ sufficiently close to λ_0 and $\varepsilon \to 0$ the function*

$$F_\varepsilon(\lambda) := -\lambda + \lambda_0 + \varepsilon^{-\alpha} \left(S_\varepsilon(\lambda) V_\varepsilon \psi_0, \psi_0 \right) \tag{14.10}$$

a) has the unique zero λ^ε which is of the first order;
b) λ^ε has the asymptotics (14.4).

Proof of Theorem 1. We construct the solution to the boundary value problem

$$\left(-\Delta + \varepsilon^{-\alpha} V_\varepsilon\right) u^\varepsilon = \lambda u^\varepsilon - f \quad \text{in} \quad \Omega, \qquad u^\varepsilon = 0, \quad \text{on} \quad \Gamma \tag{14.11}$$

as

$$u^\varepsilon = \mathcal{A}(\lambda) g_\varepsilon, \tag{14.12}$$

where $g_\varepsilon \in L_2(\Omega)$ is un unknown function. We substitute (14.12) in (14.11) and take into account the definition of the operator $\mathcal{A}(\lambda)$ to obtain the equation for the function g_ε in the domain Ω:

$$g_\varepsilon - \varepsilon^{-\alpha} V_\varepsilon \mathcal{A}(\lambda) g_\varepsilon = f.$$

Applying the operator $S_\varepsilon(\lambda)$ to this equation, we obtain

$$g_\varepsilon - \varepsilon^{-\alpha} S_\varepsilon(\lambda) V_\varepsilon \frac{(g_\varepsilon, \psi_0)}{\lambda - \lambda_0} \psi_0 = S_\varepsilon(\lambda) f. \tag{14.13}$$

We multiply the last identity by the function ψ_0 and integrate over Ω. As a result we arrive at

$$(g_\varepsilon, \psi_0) = \frac{(\lambda - \lambda_0)\,(S_\varepsilon(\lambda) f, \psi_0)}{\lambda - \lambda_0 - \varepsilon^{-\alpha}\,(S_\varepsilon(\lambda) V_\varepsilon \psi_0, \psi_0)}.$$

We substitute the obtained expression for (g_ε, ψ_0) into (14.13), which leads us to the formula for the function g_ε:

$$g_\varepsilon = S_\varepsilon(\lambda) f + \varepsilon^{-\alpha} \frac{(S_\varepsilon(\lambda) f, \psi_0)\, S_\varepsilon(\lambda) V_\varepsilon \psi_0}{\lambda - \lambda_0 - \varepsilon^{-\alpha}\,(S_\varepsilon(\lambda) V_\varepsilon \psi_0, \psi_0)}.$$

Now we substitute the last identity into (14.12) and see that the solution to the boundary value problem (14.11) reads as follows:

$$u^\varepsilon = \frac{(S_\varepsilon(\lambda) f, \psi_0)\left(\psi_0 + \widetilde{\mathcal{A}}(\lambda) S_\varepsilon(\lambda) V_\varepsilon \psi_0\right)}{\lambda - \lambda_0 - \varepsilon^{-\alpha}\,(S_\varepsilon(\lambda) V_\varepsilon \psi_0, \psi_0)} + \widetilde{\mathcal{A}}(\lambda) S_\varepsilon(\lambda) f. \tag{14.14}$$

Since the function f is arbitrary, this formula implies that the pole of the function u_ε (being an eigenvalue of the boundary value problem (14.2)) and its order coincide with the zero of the function (14.10) and its order. Now Theorem 1 follows from Lemma 1. The proof is complete.

Since the residue of the solution u^ε at the pole is the eigenfunction, it follows from (14.14) that the eigenfunction of the boundary value problem (14.2) can be represented as

$$\psi^\varepsilon = \psi_0 + \widetilde{\mathcal{A}}(\lambda^\varepsilon) S_\varepsilon(\lambda^\varepsilon) V_\varepsilon \psi_0.$$

A more detailed proof of Theorem 1 is given in [BiGa06].

14.3 The Singular-Regular Case: The Schrödinger Operator with a Large Potential Concentrated on a Small Set

The proof of Theorem 1 is essentially based on Lemma 1 for $\alpha < 1$. In this section we give the scheme of the proof of an analogue of Theorem 1 for $\alpha < 2$, and in this case it splits into two sufficiently independent parts. For simplicity, we restrict ourselves to the three-dimensional case $n = 3$.

The first part consists of the proof of the convergence of the eigenvalues of the perturbed problem (14.2) to the eigenvalues of the limiting problem (14.1) and of an estimate for the solution of the perturbed problem

$$\left(-\Delta + \varepsilon^{-\alpha} V\left(\frac{x}{\varepsilon}\right)\right) u^\varepsilon = \lambda u^\varepsilon + f_\varepsilon \quad \text{in} \quad \Omega, \quad \frac{\partial u^\varepsilon}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \Gamma, \tag{14.15}$$

for λ close to λ_0. Moreover, the solutions of the boundary value problem (14.15), of the boundary value problem

$$-\Delta u_0 = \lambda u_0 + f_0 \quad \text{in} \quad \Omega, \quad \frac{\partial u_0}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \Gamma,$$

and of the eigenvalue problems (14.1), (14.2) are treated in the weak sense. Employing the estimate (14.7) under such an approach, it is possible to prove the next statement.

Lemma 2. *Let Q be an arbitrary compact set in the complex plane $\mathbb{C}$ containing no eigenvalues of the problem (14.1). Then*

1) there exists a number $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ and any $\lambda \in Q$ there exists a unique solution u^ε to the boundary value problem (14.15);

2) if $\|f_\varepsilon - f_0\| \underset{\varepsilon \to 0}{\longrightarrow} 0$, then the convergence

$$\|u^\varepsilon - u_0\|_1 \underset{\varepsilon \to 0}{\longrightarrow} 0 \tag{14.16}$$

holds true.

This lemma allows us to prove two statements which we formulate for a simple eigenvalue of the limiting problem for the sake of brevity.

Theorem 2. *Let λ_0 be the simple eigenvalue of the boundary value problem (14.15), and ψ_0 the associated eigenfunction normalized in $L^2(\Omega)$. Then*

1) there exists a unique eigenvalue λ^ε of the boundary value problem (14.2) converging to λ_0 as $\varepsilon \to 0$, and this eigenvalue is simple;

2) for the associated eigenfunction ψ^ε normalized in $L^2(\Omega)$ the convergence $\|\psi^\varepsilon - \psi_0\|_1 \to 0$ is valid as $\varepsilon \to 0$.

Lemma 3. *Let the hypothesis of Theorem 2 hold true. Then for λ close to λ_0 the solution to the boundary value problem (14.15) satisfies a uniform in ε and λ estimate*

$$\|u^\varepsilon\|_1 \leq \frac{C}{|\lambda^\varepsilon - \lambda|} \|f_\varepsilon\|. \tag{14.17}$$

If, in addition, $(u^\varepsilon, \psi^\varepsilon) = 0$, then the uniform in ε and λ estimate

$$\|u^\varepsilon\|_1 \leq C \|f_\varepsilon\| \tag{14.18}$$

holds true.

The second part consists of constructing formal asymptotic expansions for the eigenvalue λ^ε and the eigenfunction ψ^ε by the method of matching asymptotic expansions [Il92]. Employing this method, it is possible to construct the asymptotic series

$$\lambda^\varepsilon = \lambda_0 + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \varepsilon^{\beta(i+1,j)}\lambda_{i,j}, \tag{14.19}$$

$$\lambda_{0,1} = \psi_0^2(0)\left\langle V\right\rangle, \tag{14.20}$$

$$\psi^\varepsilon(x) = \psi_0(x) + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \varepsilon^{\beta(i+1,j)}\psi_{i,j}(x), \tag{14.21}$$

$$\psi^\varepsilon(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \varepsilon^{\beta(i,j)}v_{i,j}\left(\frac{x}{\varepsilon}\right), \tag{14.22}$$

where

$$\beta(i,j) = i + (2-\alpha)j,$$

which possesses the following property.

Lemma 4. *Let $\alpha < 2$, $\chi(s)$ be an infinitely differentiable cut-off function being identically one as $s < 1$ and vanishing as $s > 2$, and let t be any fixed positive number,*

$$\begin{aligned} \lambda_N^\varepsilon &:= \lambda_0 + \sum_{i=0}^{N}\sum_{j=1}^{N} \varepsilon^{\beta(i+1,j)}\lambda_{i,j}, \\ \Psi_N^\varepsilon(x) &:= \left(1-\chi(\varepsilon^{-1/2}t|x|)\right)\left(\psi_0(x) + \sum_{i=0}^{N}\sum_{j=1}^{N}\varepsilon^{\beta(i+1,j)}\psi_{i,j}(x)\right) \\ &\qquad + \chi(\varepsilon^{-1/2}t|x|)\sum_{i=0}^{N}\sum_{j=0}^{N}\varepsilon^{\beta(i,j)}v_{i,j}\left(\frac{x}{\varepsilon}\right). \end{aligned} \tag{14.23}$$

Then

$$\|\Psi_N^\varepsilon\| = 1 + o(1) \quad \text{as } \varepsilon \to 0 \tag{14.24}$$

and the function Ψ_N^ε is a solution to the boundary value problem

$$\left(-\Delta + \varepsilon^{-\alpha}V\left(\frac{x}{\varepsilon}\right)\right)\Psi_N^\varepsilon = \lambda_N^\varepsilon\Psi_N^\varepsilon + F_N^\varepsilon \quad \text{in} \quad \Omega, \quad \frac{\partial\Psi_N^\varepsilon}{\partial\mathbf{n}} = 0 \quad \text{on} \quad \Gamma, \tag{14.25}$$

where

$$\|F_N^\varepsilon\| = O(\varepsilon^{M(N)}) \tag{14.26}$$

and $M(N)$ increases unboundedly as $N \to \infty$.

Applying the estimate (14.17) for $\lambda = \lambda_N^\varepsilon$, $f_\varepsilon = F_N^\varepsilon$, and $u^\varepsilon = \Psi_N^\varepsilon$, by the identities (14.26) and (14.24), we obtain

$$|\lambda^\varepsilon - \lambda_N^\varepsilon| = O(\varepsilon^{M(N)}). \tag{14.27}$$

Since N is arbitrary, it implies that λ^ε has the asymptotic expansion (14.19). We note that the identities (14.19), (14.20) yield also formula (14.3) as $n = 3$ but for $\alpha < 2$.

We represent Ψ_N^ε as

$$\Psi_N^\varepsilon(x) = a_N(\varepsilon)\psi^\varepsilon(x) + \psi_{\varepsilon,N}^\perp(x) \quad \text{where} \quad (\psi_{\varepsilon,N}^\perp, \psi^\varepsilon) = 0. \tag{14.28}$$

Using (14.25), we write the boundary value problem for $\psi_{\varepsilon,N}^\perp$ and employ the estimate (14.18) and the identities (14.26), (14.27), and (14.24). As a result we obtain

$$\|\psi_{\varepsilon,N}^\perp\|_1 = O(\varepsilon^{M(N)}), \qquad a_N(\varepsilon) = 1 + o(1).$$

Letting $t = 2$ in the definition (14.23) of the function Ψ_N^ε, by the identities (14.28) and the arbitrariness in the choice of N we obtain that in $\Omega \backslash \{x : |x| < \varepsilon^{1/2}\}$ the eigenfunction ψ^ε has the asymptotic expansion (14.21). By analogy, letting $t = \frac{1}{2}$ we obtain that for $|x| < 2\varepsilon^{1/2}$ the eigenfunction ψ^ε has the asymptotic expansion (14.22). In particular, it follows that for $\varepsilon^{1/2} < |x| < 2\varepsilon^{1/2}$ each of the asymptotic expansions (14.21) and (14.22) is valid.

A detailed statement is given in [Bi06].

14.4 The Regular-Singular Case: Regular Perturbation of Quantum Waveguides

In this section we consider regular perturbations of the Dirichlet boundary value problems:

$$-(\Delta + \mu_1)u_0 = -k^2 u_0 + g \quad \text{in} \quad \Pi, \qquad u_0 = 0 \quad \text{on} \quad \partial\Pi \tag{14.29}$$

in an n-dimensional cylinder $\Pi = (-\infty, \infty) \times \Omega$, where $\Omega \subset \mathbb{R}^{n-1}$ is a simply connected bounded domain with C^∞-boundary for $n \geq 3$ and is an interval (a, b) for $n = 2$. Hereinafter, μ_j and ϕ_j are the eigenvalues and eigenfunctions of $-\Delta' := -\left(\frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)$ in Ω subject to the Dirichlet boundary condition on $\partial\Omega$, $\mu_1 < \mu_2 \leqslant \cdots$. The functions ϕ_j are assumed to be normalized in $L^2(\Omega)$. It is known that unperturbed boundary value problems

$$-(\Delta + \mu_1)\psi_0 = \lambda_0\psi_0 \quad \text{in} \quad \Pi, \qquad \psi_0 = 0 \quad \text{on} \quad \partial\Pi$$

have no eigenfunctions. At the same time eigenfunctions and eigenvalues (bound states) can emerge under perturbations. Such boundary value problems are a mathematical model describing a quantum waveguide. We study the questions on the existence and absence of such emerging eigenvalues and the construction of their asymptotic expansions. The regular perturbation treated in this section is performed by a small localized linear operator of second order. An example of such an operator is a small complex potential as well as other perturbations considered in [Ga021] for the Schrödinger operator on the axis. Other examples are small deformations of strips and cylinders which can be reduced to the case we consider by a change of variables [BuGe97], [BoEx01], [DuEx95], [ExVu97].

Hereinafter $H_j^{loc}(\Pi)$ is a set of functions defined on Π whose restriction to any bounded domain $D \subset \Pi$ belongs to $H_j(D)$, and $\|\bullet\|_G$ and $\|\bullet\|_{j,G}$ are norms in $L^2(G)$ and $H_j(G)$, respectively. Next, let $Q = (-R, R) \times \Omega$, where $R > 0$ is an arbitrary fixed number, $L^2(\Pi; Q)$ be the subset of functions in $L^2(\Pi)$ with supports in $\overline{Q}$, and let $\mathcal{L}_\varepsilon$ be linear operators mapping $H_2^{loc}(\Pi)$ into $L^2(\Pi; Q)$ such that $\|\mathcal{L}_\varepsilon[u]\|_Q \leq C(\mathcal{L})\,\|u\|_{2,Q}$, where constant $C(\mathcal{L})$ is independent of ε, $0 < \varepsilon \ll 1$.

We study the existence and the asymptotics of the eigenvalues of the following Dirichlet problem:

$$-(\Delta + \mu_1 + \varepsilon\mathcal{L}_\varepsilon)\psi_\varepsilon = \lambda_\varepsilon\psi_\varepsilon \quad \text{in} \quad \Pi, \qquad \psi_\varepsilon = 0 \quad \text{on} \quad \partial\Pi. \tag{14.30}$$

For a small complex k, we define a linear operator

$$A(k) \,:\, L^2(\Pi; Q) \to H_2^{loc}(\Pi)$$

as

$$\begin{aligned} A(k)g :=& \frac{\phi_1(x')}{2k} \int\limits_{\Pi} \mathrm{e}^{-k|x_1-t_1|}\phi_1(t')g(t)\,\mathrm{d}t + \widetilde{A}(k)g, \\ \widetilde{A}(k)g :=& \sum_{j=2}^{\infty} \frac{\phi_j(x')}{2K_j(k)} \int\limits_{\Pi} \mathrm{e}^{-K_j(k)|x_1-t_1|}\phi_j(t')g(t)\,\mathrm{d}t, \end{aligned} \tag{14.31}$$

where $x' = (x_2, ..., x_n)$, and $K_j(k) = \sqrt{\mu_j - \mu_1 + k^2}$. By analogy with [Ga021] for $f \in L^2(\Pi; Q)$, we seek a solution of the boundary value problem

$$-(\Delta + \mu_1 + \varepsilon\mathcal{L}_\varepsilon)u_\varepsilon = -k^2 u_\varepsilon + f, \quad \text{in} \quad \Pi, \qquad u_\varepsilon = 0 \quad \text{on} \quad \partial\Pi \tag{14.32}$$

as

$$u_\varepsilon = A(k)g_\varepsilon, \tag{14.33}$$

where $g_\varepsilon \in L^2(\Pi; Q)$. By definition, (14.33) is the solution of the boundary value problem (14.29) for $g = g_\varepsilon$. Substituting (14.33) into (14.32), we obtain that (14.33) gives a solution for (14.32) if

$$(I - \varepsilon\mathcal{L}_\varepsilon A(k))g_\varepsilon = f, \tag{14.34}$$

where I is identity mapping.

Assume $\mathcal{L}_\varepsilon[\phi_1] \neq 0$ and denote

$$\begin{aligned} T_\varepsilon(k)g :=&\mathcal{L}_\varepsilon[A(k)g] - \frac{\langle g\phi_1\rangle}{2k}\mathcal{L}_\varepsilon[\phi_1], \\ S_\varepsilon(k) :=&\left(I - \varepsilon T_\varepsilon(k)\right)^{-1}. \end{aligned} \tag{14.35}$$

Applying the operator $S_\varepsilon(k)$ to both sides of the equation (14.34), we obtain that

$$\left(g_\varepsilon - \varepsilon\frac{\langle g_\varepsilon\phi_1\rangle}{2k}S_\varepsilon(k)\mathcal{L}_\varepsilon[\phi_1]\right) = S_\varepsilon(k)f, \tag{14.36}$$

$$\langle g_\varepsilon\phi_1\rangle\left(1 - \frac{\varepsilon}{2k}\left\langle\phi_1 S_\varepsilon(k)\mathcal{L}_\varepsilon[\phi_1]\right\rangle\right) = \left\langle\phi_1 S_\varepsilon(k)f\right\rangle. \tag{14.37}$$

The equality (14.37) allows us to determine $\langle g_\varepsilon\phi_1\rangle$. Substituting its value into (14.36), we easily get the formula

$$g_\varepsilon = \varepsilon\frac{2k\left\langle S_\varepsilon(k)f\right\rangle S_\varepsilon(k)\mathcal{L}_\varepsilon[\phi_1]}{2k - \varepsilon\left\langle\phi_1 S_\varepsilon(k)\mathcal{L}_\varepsilon[\phi_1]\right\rangle} + S_\varepsilon(k)f. \tag{14.38}$$

Formulas (14.38) and (14.33) imply that, if k_ε is a solution of the equation

$$2k - \varepsilon\left\langle\phi_1 S_\varepsilon(k)\mathcal{L}_\varepsilon[\phi_1]\right\rangle = 0, \tag{14.39}$$

then the residue of (14.33) at k_ε:

$$\psi_\varepsilon = A(k_\varepsilon)S_\varepsilon(k_\varepsilon)\mathcal{L}_\varepsilon[\phi_1] \tag{14.40}$$

is the solution of the boundary value problem (14.30), where

$$\lambda_\varepsilon = -k_\varepsilon^2. \tag{14.41}$$

Due to (14.35) the equation (14.39) has a unique small solution with the asymptotics

$$k_\varepsilon = \varepsilon\frac{1}{2}\left\langle\phi_1\mathcal{L}_\varepsilon[\phi_1]\right\rangle + O\left(\varepsilon^2\right). \tag{14.42}$$

The formulas (14.31), (14.40) yield that if $\operatorname{Re} k_\varepsilon < 0$, then $\psi_\varepsilon \notin L^2(\Pi)$ and, hence, λ_ε is not the eigenvalue, and if $\operatorname{Re} k_\varepsilon > 0$, then $\psi_\varepsilon \in L^2(\Pi)$ and, hence, λ_ε is the eigenvalue. In the last case due to (14.41) and (14.42) this eigenvalue has the asymptotics

$$\lambda_\varepsilon = -\varepsilon^2\frac{1}{4}\left\langle\phi_1\mathcal{L}_\varepsilon[\phi_1]\right\rangle^2 + O\left(\varepsilon^3\right).$$

In particular, the formula (14.42) allows us to maintain that in the case $\langle\phi_1\mathcal{L}_\varepsilon[\phi_1]\rangle \geq \delta > 0$ there exists a small eigenvalue.

If $\mathcal{L}_\varepsilon[\phi_1] = 0$, due to (14.31), (14.33), and (14.34) it follows that the pole k_ε of (14.33) is equal to zero and $g_\varepsilon \to f$ as $\varepsilon \to 0$. Thus, there is no small eigenvalue in this case.

A more detailed proof and examples are given in [Ga05].

14.5 The Twice-Singular Case: Regular Perturbation of a Quantum Waveguide

14.5.1 Convergence of Poles and Representation of a Solution Near Poles

Assume for simplicity in describing the perturbations that the domain Ω coincides with the half-space $x_n > 0$ in some neighborhood of the origin (in variables x'), ω is an $(n-1)$-dimensional bounded domain in the hyperplane $x_n = 0$ having smooth boundary, $\omega_\varepsilon = \{x : x\varepsilon^{-1} \in \omega\}$, $\Gamma_\varepsilon = \partial\Pi \backslash \overline{\omega_\varepsilon}$. For a given $f \in L^2(\Pi; Q)$, we consider the following singularly perturbed boundary value problems:

$$
\begin{aligned}
-(\Delta + \mu_1)u_\varepsilon =& - k^2 u_\varepsilon + f \quad \text{in} \quad \Pi, \\
u_\varepsilon =& 0 \quad \text{on} \quad \Gamma_\varepsilon, \quad \frac{\partial u_\varepsilon}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \omega_\varepsilon.
\end{aligned}
\tag{14.43}
$$

Let $\Gamma_0^R = \partial\Pi \cap \partial Q$, $\Omega^R = \partial Q \backslash \overline{\Gamma_0^R}$, $\Gamma_\varepsilon^R = \Gamma^R \backslash \overline{\omega_\varepsilon}$. For each $V \in H_2(Q)$, we denote by

$$
\sigma_\varepsilon : H^2(Q) \to H^1(Q)
$$

the inverse operator for the following boundary value problems:

$$
\begin{aligned}
\Delta W_\varepsilon = \Delta V \quad \text{in} \quad Q, \qquad & W_\varepsilon = V, \quad \text{on} \quad \Omega^R, \\
W_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon^R, \qquad & \frac{\partial W_\varepsilon}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \omega_\varepsilon.
\end{aligned}
$$

Let $\chi^\pm(x_1)$ be an infinitely differentiable mollifier function equalling one for $\pm x_1 \le R/2$ and vanishing for $\pm x_1 \ge R$, $\Pi_\pm = \{x : x \in \Pi, \pm x_1 > 0\}$, $p_\pm$ be the restriction operator from Π to $\Pi_\pm$, and let $p_\pm^Q$ be the restriction operator from $\Pi_\pm$ to $\Pi_\pm \cap Q$. Denote

$$
\begin{aligned}
\widetilde{A}_\pm(k)g^\pm :=& \sum_{j=2}^{\infty} \frac{\phi_j(x')}{2K_j(k)} \int\limits_{\Pi_\pm} \left(\mathrm{e}^{-K_j(k)|x_1 - t_1|} - \mathrm{e}^{-K_j(k)|x_1 + t_1|} \right) \phi_j(t') g^\pm(t) \, \mathrm{d}t, \\
A_\pm(k)g^\pm :=& \frac{\phi_1(x')}{2k} \int\limits_{\Pi_\pm} \left(\mathrm{e}^{-k|x_1 - t_1|} - \mathrm{e}^{-k|x_1 + t_1|} \right) \phi_1(t') g^\pm(t) \, \mathrm{d}t + \widetilde{A}_\pm(k) g^\pm
\end{aligned}
$$

for $x \in \Pi_\pm$, and

$$
\begin{aligned}
\mathcal{A}_\varepsilon(k)g :=& (1 - \chi^+)A_+(k)p_+ g + (1 - \chi^-)A_-(k)p_- g \\
& + \chi^+ \chi^- \sigma_\varepsilon \left(p_+^Q A_+(k) p_+ g + p_-^Q A_-(k) p_- g \right),
\end{aligned}
$$

for $g \in L^2(\Pi; Q)$.

We construct the solution of (14.43) in the form

$$u_\varepsilon = \mathcal{A}_\varepsilon^{(m)}(k) g_\varepsilon, \tag{14.44}$$

where g_ε is a function belonging to $L^2(\Pi; Q)$. Substituting (14.44) into (14.43), by analogy with [Sa80] we deduce that this function is a solution of (14.43) in the case

$$g_\varepsilon = (I + T_\varepsilon(k))^{-1} f, \tag{14.45}$$

where, for any fixed ε, $T_\varepsilon(k)$ is a holomorphic operator-valued function and, for any fixed k, $T_\varepsilon(k)$ is a compact operator in $L^2(\Pi; Q)$. An analysis of this family with respect to ε (which is similar to [Ga022] and based on [Sa80]) and the representations (14.44), (14.45) imply that there exists one pole $k_\varepsilon \to 0$ of the solution of (14.43), and for small k, this solution meets the representation

$$u_\varepsilon(x, k) = \frac{\psi_\varepsilon(x)}{2\,(k - k_\varepsilon)} \int\limits_{\Pi} \psi_\varepsilon(y)\, f(y)\, \mathrm{d}y + \widetilde{u}_\varepsilon(x, k), \tag{14.46}$$

where

$$\|\widetilde{u}_\varepsilon\|_{1,D} \leqslant C(D, Q) \|f\|_\Pi \tag{14.47}$$

for any bounded domain $D \subset \Pi$.

The residue ψ_ε at this pole is a solution to the boundary value problem

$$\begin{aligned} -(\Delta + \mu_1)\psi_\varepsilon =& \lambda_\varepsilon \psi_\varepsilon \quad \text{in} \quad \Pi, \\ \psi_\varepsilon =& 0 \quad \text{on} \quad \Gamma_\varepsilon, \quad \frac{\partial \psi_\varepsilon}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \omega_\varepsilon, \end{aligned} \tag{14.48}$$

where λ_ε defined by (14.41) and for any fixed x_1 converges to ϕ_1 as $\varepsilon \to 0$. This convergence, the representation (14.44), and the definition of $\mathcal{A}_\varepsilon(k)$ imply that

$$\psi_\varepsilon(x) = a^\varepsilon \phi_1(x') \mathrm{e}^{-|x_1| k_\varepsilon^{(m)}} + o\left(e^{-|x_1|\delta}\right) \quad \text{as } |x_1| \to \infty,$$

where $\delta > 0$ is some fixed number and

$$a^\varepsilon = 1 + o(1) \quad \text{as} \quad \varepsilon \to 0.$$

In part, these asymptotics imply that there exists eigenvalue λ_ε provided

$$\operatorname{Re} k_\varepsilon > 0. \tag{14.49}$$

Thus, in fact we need to construct and to justify asymptotics of the pole k_ε which generates the eigenvalue or does not. As mentioned above in the case of regular perturbation, the asymptotics for the pole can obtained by simple calculations in (14.39), whereas while dealing with singular perturbation, we have no such equation. On the other hand, the representation (14.46) and the estimate (14.47) allow us to justify the method of matching asymptotic expansions in constructing the asymptotics for the poles k_ε and for the residue ψ_ε.

The formal construction of complete asymptotics of poles for the boundary value problems (14.43) and for Helmholtz resonator [Ga93]–[Ga97] is similar. That is why in what follows we will construct first perturbed terms of poles only.

14.5.2 Asymptotics of Eigenvalues

Let S_n be the unit sphere in $\mathbb{R}^n$, let $G(x,y,k)$ be the Green's function of the unperturbed Dirichlet boundary value problem in Π:

$$\begin{aligned} -(\Delta+\mu_1)G(x,y,k) =& - k^2 G(x,y,k) + \delta(x-y) \quad \text{in} \quad \Pi, \\ G(x,y,k) =& 0 \quad \text{on} \quad \Gamma_\varepsilon, \qquad \frac{\partial G(x,y,k)}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \omega_\varepsilon, \end{aligned}$$

and

$$\begin{aligned} \Phi =& \frac{\partial}{\partial x_n}\phi_1(x')|_{x'=0}, \\ \Psi(x,k) =& - 2k\Phi^{-1}\frac{\partial}{\partial y_n}G(x,y,k)|_{y=0}. \end{aligned}$$

By definition $\Phi \neq 0$ and

$$\Psi(x,k) \to \phi_1(x') \quad \text{as} \quad k\to 0 \quad \text{for any fixed } x\neq 0, \tag{14.50}$$

$$\Psi(x,k) = \Phi x_n + \frac{4k}{\Phi|S_n|}\frac{x_n}{r^n} + O\left(kr^{-n+2}\right) \quad \text{as} \quad r=|x|\to 0,\, k\to 0. \tag{14.51}$$

Taking into account (14.50), outside the small neighborhood of ω_ε we construct the residue ψ_ε in the form

$$\psi_\varepsilon(x) \sim \Psi(x,k_\varepsilon).$$

Near ω_ε we construct asymptotics by using the method of matching asymptotic expansions [Il92], [Ga93]–[Ga97] in the variables $\xi = \varepsilon^{-1}x$. The structure of the expansions of ψ_ε in this zone and of the pole k_ε are inspired by the following consideration. When $x = \varepsilon\xi$ and $k = k_\varepsilon$, both terms on the right-hand side of (14.51) must have the same order with respect to ε. This degree determines the first term in the interior layer for ψ_ε, while the right-hand side of (14.51) (rewritten in variables ξ and for $k = k_\varepsilon$) determines the asymptotics of this term as $\rho = |\xi| \to \infty$. For these reasons we construct the asymptotics as

$$k_\varepsilon = \varepsilon^n\tau_n + \dots, \qquad \psi_\varepsilon(x) = \varepsilon v_1(\xi) + \dots, \tag{14.52}$$

$$v_1(\xi) = \Phi\xi_n + 4\tau_n\left(\Phi|S_n|\right)^{-1}\xi_n\rho^{-n} + o\left(\rho^{-n+1}\right), \qquad \rho\to\infty. \tag{14.53}$$

Substituting (14.52) in (14.48) for λ_ε defined by (14.41), we obtain the boundary value problem for v_1:

$$\begin{aligned} \Delta_\xi v_1 =& 0 \quad \text{for} \quad \xi_n > 0, \\ v_1 =& 0 \quad \text{on} \quad \Gamma(\omega), \qquad \frac{\partial v_1}{\partial \xi_n} = 0 \quad \text{on} \quad \omega, \end{aligned} \tag{14.54}$$

where $\Gamma(\omega) = \{\xi : \xi_n = 0,\, \xi \notin \omega\}$. It is known that there exists a solution X_n of (14.54) with asymptotics

$$X_n(\xi) = \xi_n + c_n(\omega)\xi_n\rho^{-n} + o\left(\rho^{-n+1}\right) \quad \text{as} \quad \rho \to \infty,$$

where $c_n(\omega) > 0$. Thus, it follows from (14.53) that

$$v_1(\xi) = \Phi X_n(\xi), \qquad \tau_n = \frac{1}{4}c_n(\omega)|S_n|\Phi^2 > 0. \tag{14.55}$$

By (14.52) and (14.55) we have $\operatorname{Re} k_\varepsilon > 0$ and, hence, there exists an eigenvalue (see (14.49)) and it has the asymptotics (see (14.41))

$$\lambda_\varepsilon = -\varepsilon^{2n}\left(\frac{c_n(\omega)|S_n|\Phi^2}{4}\right)^2 + o\left(\varepsilon^{2n}\right).$$

14.6 Concluding Remarks

The eigenvalues of boundary value problems are the poles of the corresponding solutions. If one treats the problems considered above as a perturbation of the poles of the solutions of these problems and their analytic continuations, then the poles exist both for the perturbed and limiting boundary value problems. Moreover, the poles of the perturbed problems converge to those of the limiting problems, as in Sections 14.2 and 14.3, or to those of the analytic continuations of the solutions of the limiting problems, as in Sections 14.4 and 14.5. So, from the point of view of the perturbation of the poles, no new poles emerge. They simply correspond to the eigenvalues in some cases and do not in others (as in Section 14.4 for $\operatorname{Re} k_\varepsilon < 0$). From this point of view, all the considered problems are regular. The same situation holds for the Helmholtz resonator and its analogues [Ga022]–[Ga97], where the pole of the solution of the limiting problem corresponds to an eigenvalue, while that of the analytic continuation of the solution of the perturbed problem does not.

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15

High-Frequency Vibrations of Systems with Concentrated Masses Along Planes

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15.1 Introduction and Statement of the Problem

Let Ω be an open bounded domain of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. We assume that Ω is divided into two parts Ω_+ and Ω_- by the plane γ: $\Omega = \Omega_+ \cup \Omega_- \cup \gamma$. For simplicity, we assume that the plane $\{x_3 = 0\}$ cuts Ω and $\gamma = \Omega \cap \{x_3 = 0\}$. Let ε be a small positive parameter that tends to zero. We denote by ω_ε the ε-neighborhood of γ, i.e., $\omega_\varepsilon = \Omega \cap \{|x_3| < \varepsilon\}$; for ε sufficiently small, we assume that $\omega_\varepsilon = \gamma \times (-\varepsilon, \varepsilon)$ (see Figure 15.1). Note that this conditions the geometry of Ω near γ. Let us denote by $\bar{x}$ the two first components of any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, that is, $\bar{x} = (x_1, x_2)$.

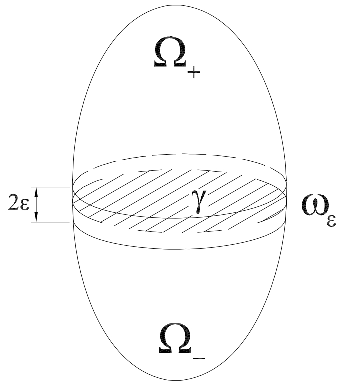

Fig. 15.1. A geometrical configuration.

We consider the eigenvalue problem

$$\begin{cases} -\Delta u^\varepsilon = \lambda^\varepsilon \rho_\varepsilon u^\varepsilon & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega \end{cases}, \qquad (15.1)$$

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where ρ_ε is the density function

$$\rho_\varepsilon(x) = \begin{cases} p & \text{if } x \in \Omega \setminus \overline{\omega_\varepsilon} \\ q\varepsilon^{-m} & \text{if } x \in \omega_\varepsilon \end{cases}$$

for m a positive parameter, and p and q positive constants. We assume $m > 1$. The spectral problem deals with the vibrations of a system composed of a body that contains a thin region where the density is much higher than elsewhere, the so-called *concentrated mass along planes*: the size and the density of the region ω_ε are of order $O(\varepsilon)$ and $O(\varepsilon^{-m})$, respectively, while they are of order $O(1)$ outside.

The variational formulation of (15.1) is: Find λ^ε and $u^\varepsilon \in H_0^1(\Omega)$, $u^\varepsilon \neq 0$, satisfying

$$\int_\Omega \nabla u^\varepsilon \cdot \nabla v \, dx = \lambda^\varepsilon \left[\int_{\Omega \setminus \overline{\omega_\varepsilon}} p u^\varepsilon v \, dx + \frac{1}{\varepsilon^m} \int_{\omega_\varepsilon} q u^\varepsilon v \, dx \right], \quad \forall v \in H_0^1(\Omega). \tag{15.2}$$

For each fixed $\varepsilon > 0$, problem (15.2) is a standard eigenvalue problem in $H_0^1(\Omega)$. Let us consider $\{\lambda_i^\varepsilon\}_{i=1}^\infty$ the sequence of eigenvalues of (15.2), with the classical convention of repeated eigenvalues. Let $\{u_i^\varepsilon\}_{i=1}^\infty$ be the corresponding eigenfunctions, which form an orthonormal basis in $H_0^1(\Omega)$, that is,

$$\int_\Omega \nabla u_i^\varepsilon \cdot \nabla u_j^\varepsilon \, dx = \delta_{i,j} \quad \text{for } i, j = 1, 2 \ldots . \tag{15.3}$$

The aim of this chapter is to study the asymptotic behavior of certain eigenelements $(\lambda^\varepsilon, u^\varepsilon)$ of (15.1) as $\varepsilon \to 0$.

15.1.1 Preliminary Results

Many authors have addressed the asymptotic behavior of vibrating systems with concentrated masses at points (cf. [LoPe03] for references), but only a few of them consider vibrating systems with concentrated masses on manifolds. See [GoGo02] and [GoGo04] for the vibrations of a membrane with a concentrated mass around a curve; [GoLo06] for problems with stiff regions and concentrated masses along curves where very different techniques are used. For dimension three, the only references are [Tc84], for $m = 1$, and [GoLo05], for $m > 1$, regarding the low frequencies.

First, we introduce two inequalities which will be useful throughout the chapter:

$$\int_{\omega_\varepsilon} |u|^2 \, dx \leq C\varepsilon \|u\|^2_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega) \tag{15.4}$$

and

$$\left| \frac{1}{\varepsilon} \int_{\omega_\varepsilon} uv \, dx - 2 \int_\gamma uv \, d\bar{x} \right| \leq C\varepsilon^{1/2} \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}, \quad \forall u, v \in H^1(\Omega), \tag{15.5}$$

where C is a constant independent of ε, u, and v. Let us refer to [MaHr74] for the proof of (15.4), whereas the inequality (15.5) holds from (15.4) (see also [Tc84]).

We obtain the following bound for the eigenvalues of (15.1).

Lemma 1 *For each fixed $i = 1, 2, 3 \ldots$, and ε sufficiently small, we have*

$$C\varepsilon^{m-1} \leq \lambda_i^\varepsilon \leq C_i \varepsilon^{m-1} \quad \textit{for } m > 1 \tag{15.6}$$

where C, C_i are constants independent of ε and $C_i \to \infty$ when $i \to \infty$.

Proof. The left-hand side of (15.6) holds easily from the variational formulation (15.2), the Poincaré inequality, and (15.4), that is,

$$\begin{aligned} \lambda_i^\varepsilon &\geq \frac{\int_\Omega |\nabla u_i^\varepsilon|^2 \, dx}{\int_\Omega p |u_i^\varepsilon|^2 \, dx + \varepsilon^{-m} \int_{\omega_\varepsilon} q |u_i^\varepsilon|^2 \, dx} \\ &\geq \frac{\int_\Omega |\nabla u_i^\varepsilon|^2 \, dx}{K_1 \int_\Omega |\nabla u_i^\varepsilon|^2 \, dx + \varepsilon^{1-m} K_2 \int_\Omega |\nabla u_i^\varepsilon|^2 \, dx} \geq C\varepsilon^{m-1}, \end{aligned}$$

where K_1, K_2, C are constants independent of ε.

On the other hand, the minimax principle gives the equality

$$\lambda_i^\varepsilon = \min_{\substack{E_i \subset H_0^1(\Omega) \\ \dim E_i = i}} \max_{\substack{v \in E_i \\ v \neq 0}} \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_{\Omega \setminus \overline{\omega_\varepsilon}} p |v|^2 \, dx + \varepsilon^{-m} \int_{\omega_\varepsilon} q |v|^2 \, dx}, \tag{15.7}$$

where the minimum is taken over all the subspaces $E_i \subset H_0^1(\Omega)$ with $\dim E_i = i$. For each fixed i, let us consider E_i^* the subspace of $H_0^1(\Omega)$, $E_i^* = [u_1, \ldots, u_i]$, where $\{u_i\}_{i=1}^\infty$ are the eigenfunctions of (15.10) which are assumed to be orthonormal in $H_0^1(\Omega)$. Then, taking in (15.7) the particular subspace E_i^*, we obtain

$$\lambda_i^\varepsilon \leq \max_{\substack{v \in E_i^* \\ v \neq 0}} \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_{\Omega \setminus \overline{\omega_\varepsilon}} p |v|^2 \, dx + \varepsilon^{-m} \int_{\omega_\varepsilon} q |v|^2 \, dx} \leq \max_{\substack{v \in E_i^* \\ v \neq 0}} \frac{\int_\Omega |\nabla v|^2 \, dx}{\varepsilon^{-m} \int_{\omega_\varepsilon} q |v|^2 \, dx}. \tag{15.8}$$

From the orthogonality condition of the eigenfunctions u_i in $H_0^1(\Omega)$ and in $L^2(\gamma)$, we have

$$2q \int_\gamma |v|^2 \, d\gamma \geq \frac{1}{\lambda_i} \|\nabla v\|_{L^2(\Omega)}^2, \quad \forall v \in E_i^*,$$

and, using (15.5), we get

$$\frac{1}{\varepsilon} \int_{\omega_\varepsilon} q |v|^2 \, dx \geq \frac{1}{\lambda^i} \|\nabla v\|_{L^2(\Omega)}^2 - C\varepsilon^{1/2} \|\nabla v\|_{L^2(\Omega)}^2, \quad \forall v \in E_i^*.$$

Now, by introducing this inequality in (15.8) we obtain the right-hand side of (15.6) and the lemma is proved.

Estimate (15.6) allows us to state the spectral concentration phenomena at the origin for the low frequencies, a rescaling being necessary to detect their asymptotic behavior. Theorem 1 in Section 15.2 characterizes this behavior via the eigenelements of the problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_+ \cup \Omega_-, \\ [u] = 0, \ \left[\dfrac{\partial u}{\partial x_3}\right] + \lambda\, 2qu = 0 & \text{in } \gamma, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{15.9}$$

where the brackets mean the jump of the enclosed quantities across γ, that is, $[v] = v(\bar{x}, 0^+) - v(\bar{x}, 0^-)$ for $\bar{x} \in \gamma$.

As happens in other vibrating systems with concentrated masses, for $m > 1$, the *high frequencies*, namely the eigenvalues $\lambda^\varepsilon = O(\varepsilon^\alpha)$ with $\alpha < m-1$, accumulate in the whole positive real axis $[0, \infty)$. We refer to [GoLo99] for a proof of this result using spectral families for systems with a concentrated mass at a point. See [CaZu96] for a general result for self-adjoint and compact operators, where ε ranges in certain subsequences. For brevity, on account of the results of Section 15.2, in Section 15.3, we use the result in [CaZu96] to prove the existence of converging sequences of eigenvalues $\lambda^\varepsilon_{i(\varepsilon)}$ for problem (15.1) with $m > 1$. See [GoGo04] for the results related to a vibrating membrane with a concentrated mass along a curve.

Depending on the value of $m > 1$, there are different behaviors of these eigenvalues of higher order and their corresponding eigenfunctions. For brevity, here we provide the detailed proof for $m = 3$ and the frequencies of order $O(1)$: see Theorems 2, 3, and 4 in Section 15.3. The limiting problem for these high frequencies was outlined in [GoLo07] without any proof. We leave the asymptotics for the so-called *middle frequencies* for a forthcoming publication.

For completeness, we summarize in Section 15.2 the results for the low frequencies; see [GoLo05] for details.

15.2 Low Frequencies

In this section, we address the asymptotic behavior, as $\varepsilon \to 0$, of the eigenvalues λ_i^ε of problem (15.1) for i fixed and of the corresponding eigenfunctions u_i^ε.

For $m > 1$, let us assume the asymptotic expansions

$$\lambda_i^\varepsilon = \lambda_i \varepsilon^{m-1} + o(\varepsilon^{m-1}) \quad \text{and} \quad u_i^\varepsilon = u_i + o(1) \text{ in } H_0^1(\Omega) - \text{weak},$$

for some real λ_i and some function in $H_0^1(\Omega)$ u_i, $u_i \neq 0$. Then, taking limits in the variational formulation (15.2) when $\varepsilon \to 0$ and using (15.5), we can identify (λ_i, u_i) as an eigenelement of the following problem: Find λ and $u \in H_0^1(\Omega)$, $u \neq 0$, such that

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \lambda \, 2 \int_{\gamma} quv \, d\bar{x}, \quad \forall v \in H_0^1(\Omega), \tag{15.10}$$

which is the integral formulation of the Steklov-type eigenvalue problem (15.9).

Problem (15.10) has a real, positive, and discrete spectrum. Let us denote by $\{\lambda_i\}_{i=1}^{\infty}$ the sequence of eigenvalues of (15.10), with the usual convention of repeated eigenvalues, and by $\{u_i\}_{i=1}^{\infty}$ the associated eigenfunctions.

Theorem 1 states the convergence of the eigenvalues λ_i^{ε} of (15.2) and their corresponding eigenfunctions. Previously, we introduce some operators associated with problems (15.2) and (15.10).

Let $H_{\varepsilon} = H$ be the space $H_0^1(\Omega)$. Let us consider A_{ε} the positive, self-adjoint and compact operator defined on H_{ε} by $A_{\varepsilon} f = u^{\varepsilon}$, where $u^{\varepsilon} \in H_0^1(\Omega)$ is the unique solution of

$$\int_{\Omega} \nabla u^{\varepsilon} \cdot \nabla v \, dx = \varepsilon^{m-1} \int_{\Omega \setminus \overline{\omega_{\varepsilon}}} pfv \, dx + \varepsilon^{-1} \int_{\omega_{\varepsilon}} qfv \, dx, \quad \forall v \in H_0^1(\Omega). \tag{15.11}$$

The eigenvalues of A_{ε} are $\{\varepsilon^{m-1}/\lambda_i^{\varepsilon}\}_{i=1}^{\infty}$, where $\{\lambda_i^{\varepsilon}\}_{i=1}^{\infty}$ are the eigenvalues of (15.2).

In the same way, we consider A the self-adjoint and compact operator defined on H by $Af = u$, where $u \in H_0^1(\Omega)$ is the unique solution of

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = 2 \int_{\gamma} qfv \, d\bar{x}, \quad \forall v \in H_0^1(\Omega). \tag{15.12}$$

The eigenvalues of A are $\{1/\lambda_i\}_{i=1}^{\infty} \cup \{0\}$, where $\{\lambda_i\}_{i=1}^{\infty}$ are the eigenvalues of (15.10) with finite multiplicity, whereas $\lambda = 0$ is an eigenvalue of infinite multiplicity; the eigenspace associated with $\lambda = 0$ is $W = \{v \in H_0^1(\Omega) : v = 0 \text{ on } \gamma\}$.

Let H_0 be the orthogonal complement of W in $H_0^1(\Omega)$ and let R_{ε} be the identity operator from H_0 to H_{ε}. By definition of the operator A, $ImA \subset H_0$; we consider $A_0 : H_0 \to H_0$ the restriction operator of A. Now, A_0 is a positive, self-adjoint, and compact operator whose eigenvalues are $\{1/\lambda_i\}_{i=1}^{\infty}$, where $\{\lambda_i\}_{i=1}^{\infty}$ are the eigenvalues of (15.10).

Theorem 1. *Let λ_i^{ε} be the eigenvalues of problem (15.2) and u_i^{ε} the corresponding eigenfunctions such that $\|\nabla u_i^{\varepsilon}\|_{L^2(\Omega)} = 1$. If $m > 1$, for each i fixed, the sequence $\lambda_i^{\varepsilon}/\varepsilon^{m-1}$ converges, when $\varepsilon \to 0$, towards λ_i, the ith eigenvalue of (15.10). Moreover, for any eigenvalue λ_i of (15.10) with multiplicity $\varkappa$ ($\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+\varkappa-1}$) and for any eigenfunction u of (15.10) associated with λ_i such that $\|\nabla u\|_{L^2(\Omega)} = 1$, there exists a linear combination $\tilde{u}^{\varepsilon}$ of eigenfunctions associated with $\{\lambda_k^{\varepsilon}\}_{k=i}^{i+\varkappa-1}$ such that $\tilde{u}^{\varepsilon}$ converges towards u in $H^1(\Omega)$.*

In addition, for each sequence u_i^{ε} we can extract a subsequence, still denoted by ε, such that u_i^{ε} converges towards u_i^* in $H_0^1(\Omega)$, where u_i^* is an eigenfunction of (15.10) associated with λ_i, and $\{u_i^*\}_{i=1}^{\infty}$ form an orthonormal basis in the orthogonal complement of $\{v \in H_0^1(\Omega) : v|_{\gamma} = 0\}$ in $H_0^1(\Omega)$.

Sketch of the proof. For each $\varepsilon > 0$ and fixed $f \in H_0$, we consider $u^\varepsilon = A_\varepsilon R_\varepsilon f$; $u^\varepsilon \in H_0^1(\Omega)$ verifies (15.11). Taking limits in (15.11) and using (15.5) we obtain that u^ε converges to u^* strongly in $H_0^1(\Omega)$ when $\varepsilon \to 0$, where u^* verifies (15.12), that is, $u^* = A_0 f$. Thus, applying the spectral convergence theorem for positive, symmetric, and compact operators on a varying Hilbert space (cf. Section III.1 in [OlSh92]), the convergence of the eigenvalues holds as the theorem states.

As regards the proof of the last statement in the theorem, we refer to [GoLo05] for further details.

Remark 1. The above theorem is related to the low frequencies of (15.1) for $m > 1$. Our technique also applies to the case $0 < m \leq 1$; then, the eigenvalues λ_i^ε are of order $O(1)$ (cf. Lemma 1) and the limiting problem is different (see [Tc84] for the case $m = 1$ and different techniques).

15.3 Frequencies of Higher Order

The aim of this section is to study the asymptotic behavior, as $\varepsilon \to 0$, of the eigenvalues of (15.1) of higher order than $O(\varepsilon^{m-1})$ for $m > 1$; that is, converging sequences $\lambda_{i(\varepsilon)}^\varepsilon$ of order $O(\varepsilon^\alpha)$ for $\alpha < m - 1$. In particular, we focus on the asymptotic behavior of the eigenvalues λ^ε of order $O(1)$ of problem (15.1) for $m = 3$ and of the corresponding eigenfunctions u^ε.

For completeness, we first introduce two general results for self-adjoint and compact operators. Lemma 2 is related to the spectral convergence for large frequencies (see [CaZu96] for the proof). Lemma 3 is related to "almost eigenvalues and eigenfunctions" from the spectral perturbations theory; we refer to Section III.1 in [OlSh92] for the proof.

Lemma 2 *Let $\{T^\varepsilon\}_{\varepsilon \in [0,1]}$ be a family of self-adjoint and compact operators on a Hilbert space H. For each ε, let $\{\mu_i^\varepsilon\}_{i=1}^\infty$ be the sequence of the eigenvalues of T^ε with the classical convention of repeated eigenvalues. Let us assume that the family T^ε satisfies the following property: for each $i \in \mathbb{N}$ the function $\mu^i(\varepsilon) = \mu_i^\varepsilon$ is continuous with respect to ε in $[0,1]$. Then, for each $\beta > 0$ and $\lambda > 0$ there exists a sequence $\varepsilon_j \to 0$ and a sequence of natural numbers $\{i(\varepsilon_j)\}_{j \in \mathbb{N}}$, $i(\varepsilon_j) \to \infty$, such that $\left(\mu_{i(\varepsilon_j)}^{\varepsilon_j}\right)^{-1} \varepsilon_j^\beta = \lambda$.*

Lemma 3 *Let $A : H \longrightarrow H$ be a linear, self-adjoint, positive, and compact operator on a Hilbert space H. Let $u \in H$, with $\|u\|_H = 1$ and λ, $r > 0$ such that $\|Au - \lambda u\|_H \leq r$. Then, there exists an eigenvalue λ_i of A satisfying $|\lambda - \lambda_i| \leq r$. Moreover, for any $r^* > r$ there is $u^* \in H$, with $\|u^*\|_H = 1$, u^* belonging to the eigenspace associated with all the eigenvalues of operator A lying on the segment $[\lambda - r^*, \lambda + r^*]$, such that*

$$\|u - u^*\|_H \leq \frac{2r}{r^*}.$$

We observe that the operators A_ε for $\varepsilon \in (0,1)$ and A, defined in Section 15.2, verify the conditions of Lemma 2 for $\mu_i^\varepsilon = \varepsilon^{m-1}/\lambda_i^\varepsilon$ with λ_i^ε eigenvalues of (15.1). Thus, if $m > 1$, for each $\alpha < m-1$ and $\lambda > 0$, there exists a sequence $\varepsilon_j \to 0$ and a sequence of natural numbers $\{i(\varepsilon_j)\}_{j\in\mathbb{N}}$, $i(\varepsilon_j) \to \infty$, such that

$$\lambda_{i(\varepsilon_j)}^{\varepsilon_j}/\varepsilon_j^\alpha = \lambda.$$

In particular, if $\alpha = 0$, we have that for any $\lambda > 0$ there exists a subsequence of eigenvalues $\lambda_{i(\varepsilon_j)}^{\varepsilon_j}$ of (15.1) converging towards λ as $\varepsilon_j \to 0$. For simplicity, we still denote by ε this subsequence.

As we verify in Section 15.3.1 for $\alpha = 0$, for certain sequences of eigenvalues $\lambda_{i(\varepsilon)}^\varepsilon = O(\varepsilon^\alpha)$ with $\alpha < m-1$, there is a different behavior of the corresponding eigenfunctions according to whether the values $\lambda_{i(\varepsilon)}^\varepsilon/\varepsilon^\alpha$ are asymptotically near eigenvalues of certain problems or not. In fact, for $m = 3$, different limit behaviors are obtained for the eigenfunctions associated with the eigenvalues $\lambda^\varepsilon = O(\varepsilon)$ and $\lambda^\varepsilon = O(1)$. In the rest of the section, we address the asymptotic behavior of the eigenfunctions u^ε associated with eigenvalues $\lambda_{i(\varepsilon)}^\varepsilon$ of order $O(1)$ under the assumption that the eigenfunctions u^ε satisfy (15.3).

15.3.1 The Case $m = 3$ and Frequencies of Order $O(1)$

Let us first proceed formally. We consider the asymptotic expansions

$$\lambda^\varepsilon = \lambda + o(1) \quad \text{and} \quad u^\varepsilon = u + o(1) \text{ in } H_0^1(\Omega)$$

with $\lambda \neq 0$. Then, replacing these expansions in (15.1), we get the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda p u & \text{in } \Omega_+ \cup \Omega_-, \\ u = 0 & \text{in } \partial\Omega \cup \gamma. \end{cases} \tag{15.13}$$

We notice a different behavior of the eigenfunctions associated with eigenvalues λ^ε of order $O(1)$ depending on whether they are close to eigenvalues of (15.13) or not. Next, we state the convergence results describing this behavior.

Theorem 2. *Let λ_i^ε be the eigenvalues of (15.1) and u_i^ε the corresponding eigenfunctions such that $\|\nabla u_i^\varepsilon\|_{L^2(\Omega)} = 1$. Let us assume that $\lambda_{i(\varepsilon)}^\varepsilon \to \lambda$, as $\varepsilon \to 0$, and the corresponding eigenfunctions $u_{i(\varepsilon)}^\varepsilon$ converge towards u in $H_0^1(\Omega)$-weak.*

i) If $u \neq 0$, then (λ, u) is an eigenelement of (15.13).
ii) If (λ, u) is not an eigenelement of (15.13), then $u_{i(\varepsilon)}^\varepsilon$ converge towards 0 in $L^2(\Omega)$ as $\varepsilon \to 0$.

Proof. First, we prove that $u = 0$ on γ. Since $(\lambda^\varepsilon, u^\varepsilon)$ satisfy (15.2) and λ^ε and $\|\nabla u^\varepsilon\|_{L^2(\Omega)}$ are bounded, it follows that

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u^\varepsilon|^2\,dx \le C\varepsilon^2, \tag{15.14}$$

where C is a constant independent of ε. Then, using (15.5) and the weak convergence in $H_0^1(\Omega)$ of u^ε towards u as $\varepsilon \to 0$ yields that u vanishes on γ.

Thus, if $u \neq 0$ in Ω_+ (Ω_- resp.), taking limits in (15.2) for $v \in \mathcal{D}(\Omega_+)$ ($v \in \mathcal{D}(\Omega_-)$ resp.), we obtain that (λ, u) is an eigenelement of the Dirichlet problem in Ω_+ (Ω_- resp.) and statement *i)* is proved.

Statement *ii)* holds by contradiction, and the proof is complete.

Theorem 3. *Let us consider $\lambda > 0$ and $\{d^\varepsilon\}_\varepsilon$ any sequence such that $d^\varepsilon \to 0$ as $\varepsilon \to 0$. Let $\{\lambda^\varepsilon_{i(\varepsilon)}, \lambda^\varepsilon_{i(\varepsilon)+1}, \dots, \lambda^\varepsilon_{i(\varepsilon)+k(\varepsilon)}\}$ be all the eigenvalues of (15.1) in the interval $[\lambda - d^\varepsilon, \lambda + d^\varepsilon]$, and $\tilde{u}^\varepsilon$ any function in the eigenspace $[u^\varepsilon_{i(\varepsilon)}, u^\varepsilon_{i(\varepsilon)+1}, \dots, u^\varepsilon_{i(\varepsilon)+k(\varepsilon)}]$ with $\|\nabla \tilde{u}^\varepsilon\|_{L^2(\Omega)} = 1$.*

i) If there is some converging subsequence $\{\tilde{u}^{\varepsilon_k}\}_k$ such that $\|\tilde{u}^{\varepsilon_k}\|_{L^2(\Omega)} > a > 0$, for some constant a independent of ε_k, then (λ, u^*) is an eigenelement of (15.13), where u^* is the limit of $\{\tilde{u}^{\varepsilon_k}\}_k$ as $\varepsilon_k \to 0$.
ii) If λ is not an eigenvalue of (15.13), then $\tilde{u}^\varepsilon$ converge towards 0 in $L^2(\Omega)$ as $\varepsilon \to 0$.

The assertions hold in view of Theorem 2 with minor modifications (see [GoLo99] for the technique).

Theorem 4. *Let λ be an eigenvalue of problem (15.13) and u an associated eigenfunction such that $\|\nabla u\|_{L^2(\Omega)} = 1$. Then, there are eigenvalues $\lambda^\varepsilon_{i(\varepsilon)}$ of problem (15.1) such that*

$$|\lambda^\varepsilon_{i(\varepsilon)} - \lambda| \le C\varepsilon^{1/2}, \tag{15.15}$$

where C is a constant independent of ε. Moreover, there is a linear combination $\tilde{u}^\varepsilon \in H_0^1(\Omega)$ of eigenfunctions associated with the eigenvalues $\lambda^\varepsilon_{i(\varepsilon)}$ of (15.1) which satisfy $\lambda^\varepsilon_{i(\varepsilon)} \in [\lambda - K\varepsilon^\theta, \lambda + K\varepsilon^\theta]$ with $K > 0$ and $0 < \theta < 1/2$, $\|\nabla \tilde{u}^\varepsilon\|_{L^2(\Omega)} = 1$, such that

$$\|\nabla(\tilde{u}^\varepsilon - \alpha^\varepsilon \varphi^\varepsilon u)\|_{L^2(\Omega)} \le C\varepsilon^{1/2-\theta}, \tag{15.16}$$

where C is a constant independent of ε, $\varphi^\varepsilon(x) = \varphi(x_3/\varepsilon)$ with $\varphi \in C^\infty(\mathbb{R})$, $0 \le \varphi \le 1$, $\varphi(r) = 0$ if $|r| \le 1$ and $\varphi(r) = 1$ if $|r| \ge 2$, and α^ε is a sequence of constants that converges to 1 as $\varepsilon \to 0$.

Proof. Let V_ε be the space $H_0^1(\Omega)$ with the scalar product

$$(v, w)_\varepsilon = \int_\Omega \nabla v \cdot \nabla w\,dx + \int_{\Omega \setminus \overline{\omega_\varepsilon}} pvw\,dx + \varepsilon^{-3}\int_{\omega_\varepsilon} qvw\,dx, \quad \forall v, w \in H_0^1(\Omega).$$

Let us consider B_ε the operator defined on V_ε by

$$(B_\varepsilon v, w)_\varepsilon = \int_{\Omega\setminus\overline{\omega_\varepsilon}} pvw\,dx + \varepsilon^{-3}\int_{\omega_\varepsilon} qvw\,dx, \quad \forall v, w \in V_\varepsilon.$$

Obviously, B_ε is a positive, self-adjoint, and compact operator whose eigenvalues are $\{1/(\lambda_i^\varepsilon + 1)\}_{i=1}^\infty$, where $\{\lambda_i^\varepsilon\}_{i=1}^\infty$ are the eigenvalues of (15.2).

Let (λ, u) be an eigenelement of (15.13). Let us prove that $u \in H^2(\Omega_\pm)$. Indeed, if $u \neq 0$ in Ω_+, let us consider the domain $\Omega_+^0 = \{(\bar{x}, x_3) : (\bar{x}, x_3) \in \Omega_+ \text{ or } (\bar{x}, -x_3) \in \Omega_+\} \cup \gamma$ and the function $\tilde{u}(x) = u(x)$ if $x \in \overline{\Omega_+}$ and $\tilde{u}(x) = -u(\bar{x}, -x_3)$ if $(\bar{x}, -x_3) \in \Omega_+$. Because of the geometry of Ω, the boundary of Ω_+^0 is smooth. Besides, we can check that $(\lambda, \tilde{u})$ is an eigenelement of the Dirichlet problem in Ω_+^0. In fact, from the definition of $\tilde{u}$, it is clear that $-\Delta\tilde{u} = \lambda\tilde{u}$ in Ω_+ and $\Omega_+^0 \cap \{x_3 < 0\}$. Moreover, since $\tilde{u}$ vanishes on γ, applying the Green formula, we have that for any $\psi \in \mathcal{D}(\Omega_+^0)$,

$$\begin{aligned}< -\Delta\tilde{u}, \psi >_{\mathcal{D}'(\Omega_+^0)\mathcal{D}(\Omega_+^0)} &= -\int_{\Omega_+^0} \tilde{u}\Delta\psi\,dx = \int_{\Omega_+^0} \nabla\tilde{u}\cdot\nabla\psi\,dx \\ &= -\int_{\Omega_+} \Delta\tilde{u}\psi\,dx - \int_{\Omega_+^0\cap\{x_3<0\}} \Delta\tilde{u}\psi\,dx - \int_\gamma \left(\frac{\partial\tilde{u}}{\partial x_3}\bigg|_{\gamma_+} - \frac{\partial\tilde{u}}{\partial x_3}\bigg|_{\gamma_-}\right)\psi\,d\bar{x},\end{aligned} \tag{15.17}$$

where $v|_{\gamma_\pm}$ denote $v(\bar{x}, 0^\pm)$, respectively (cf. Section III.9 of [SaSa89]). Now, using again the definition of $\tilde{u}$, the last term in the above expression is zero and $-\Delta\tilde{u} = \lambda\tilde{u}$ in Ω_+^0 in the sense of distributions. Thus, $(\lambda, \tilde{u})$ is an eigenelement of the Dirichlet problem in Ω_+^0; consequently, $\tilde{u} \in H^2(\Omega_+^0)$ and $u \in H^2(\Omega_+)$. The same proof holds for Ω_-.

Let u and φ^ε be as the theorem states. It is clear that $\varphi^\varepsilon u \in H_0^1(\Omega)$ and

$$\|\nabla(u - \varphi^\varepsilon u)\|_{L^2(\Omega)}^2 \leq C_1\|\nabla u\|_{L^2(\omega_{2\varepsilon})}^2 + C_2\varepsilon^{-2}\|u\|_{L^2(\omega_{2\varepsilon})}^2,$$

where C_1 and C_2 are constants independent of ε and $\omega_{2\varepsilon}$ denotes the 2ε-neighborhood of γ. As u vanishes on γ, using the techniques to derive (15.5), we obtain

$$\|u\|_{L^2(\omega_{2\varepsilon})}^2 \leq C\varepsilon^2\|\nabla u\|_{L^2(\omega_{2\varepsilon})}^2.$$

Also, since $u \in H^2(\Omega_\pm)$, (15.4) gives

$$\|\nabla u\|_{L^2(\omega_{2\varepsilon}^\pm)}^2 \leq C\varepsilon\|u\|_{H^2(\Omega_\pm)}^2,$$

where $\omega_{2\varepsilon}^\pm$ denote $\omega_{2\varepsilon} \cap \Omega_\pm$, respectively. Hence,

$$\|\nabla(u - \varphi^\varepsilon u)\|_{L^2(\Omega)} \leq C\varepsilon^{1/2}, \tag{15.18}$$

with C a constant independent of ε. Let us note that

$$\|\varphi^\varepsilon u\|_\varepsilon^2 \to \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L_p^2(\Omega)}^2 = 1 + \frac{1}{\lambda} \quad \text{as } \varepsilon \to 0.$$

In order to apply Lemma 3, we prove the following estimate:

$$\left|\left(B_\varepsilon v^\varepsilon - \frac{1}{1+\lambda} v^\varepsilon, w\right)_\varepsilon\right| \leq C\varepsilon^{1/2}\|w\|_\varepsilon, \quad \forall w \in V_\varepsilon, \tag{15.19}$$

where $v^\varepsilon = \varphi^\varepsilon u / \|\varphi^\varepsilon u\|_\varepsilon$.

Because of the definition of B_ε, $(\cdot,\cdot)_\varepsilon$, φ^ε, and the fact that (λ, u) is an eigenelement of (15.13), we can write

$$(1+\lambda) \quad \left|(B_\varepsilon(\varphi^\varepsilon u) - \frac{1}{1+\lambda}\varphi^\varepsilon u, w)_\varepsilon\right| = \lambda \int_\Omega p(\varphi^\varepsilon u - u) w\, dx$$
$$- \int_\Omega \nabla(\varphi^\varepsilon u - u)\cdot \nabla w\, dx + \int_\gamma \left(\left.\frac{\partial u}{\partial x_3}\right|_{\gamma_+} - \left.\frac{\partial u}{\partial x_3}\right|_{\gamma_-}\right) w\, d\bar{x} \tag{15.20}$$

for all $w \in H_0^1(\Omega)$. Besides, since $u \in H^2(\Omega_\pm)$, it can be verified [see the technique for (15.5)]

$$\left| \int_\gamma \left.\frac{\partial u}{\partial x_3}\right|_{\gamma_\pm} w\, d\bar{x} - \frac{1}{\varepsilon}\int_{\omega_\varepsilon^\pm} \frac{\partial u}{\partial x_3} w dx \right|$$
$$\leq C \left[\left\| \frac{\partial^2 u}{\partial x_3^2} \right\|_{L^2(\omega_\varepsilon^\pm)} \|w\|_{L^2(\omega_\varepsilon^\pm)} + \left\| \frac{\partial u}{\partial x_3}\right\|_{L^2(\omega_\varepsilon^\pm)} \left\| \frac{\partial w}{\partial x_3}\right\|_{L^2(\omega_\varepsilon^\pm)} \right]$$

with C a constant independent of ε, and, because of (15.4) and the definition of $(\cdot,\cdot)_\varepsilon$, we get

$$\left| \int_\gamma \left.\frac{\partial u}{\partial x_3}\right|_{\gamma_\pm} w\, d\bar{x} \right| \leq C\varepsilon^{1/2}\|w\|_\varepsilon, \quad \forall w \in H_0^1(\Omega). \tag{15.21}$$

Thus, from (15.20), (15.18), and (15.21), (15.19) holds.

We apply Lemma 3 with $A = B_\varepsilon$, $H = V_\varepsilon$, $u = v^\varepsilon$, and $r = C\varepsilon^{1/2}$, and we deduce that there exist eigenvalues $\lambda^\varepsilon_{i(\varepsilon)}$ of (15.1) verifying $|1/(\lambda^\varepsilon_{i(\varepsilon)}+1) - 1/(\lambda+1)| \leq C\varepsilon^{1/2}$, and, since $\lambda^\varepsilon_{i(\varepsilon)}$ are bounded by a constant independent of ε, (15.15) is true.

In addition, if we take $r^* = \varepsilon^\theta$ with $0 < \theta < 1/2$, Lemma 3 ensures the existence of $\tilde{v}^\varepsilon$, with $\|\tilde{v}^\varepsilon\|_\varepsilon = 1$, $\tilde{v}^\varepsilon$ belonging to the eigenspace associated with all the eigenvalues of operator B_ε lying on the segment $[1/(\lambda+1)-\varepsilon^\theta, 1/(\lambda+1)+\varepsilon^\theta]$, such that $\|\tilde{v}^\varepsilon - v^\varepsilon\|_\varepsilon \leq C\varepsilon^{1/2-\theta}$. Then, $\|\nabla\tilde{v}^\varepsilon\|^2_{L^2(\Omega)} \to 1/(1+1/\lambda)$, and (15.16) holds for $\tilde{u}^\varepsilon = \tilde{v}^\varepsilon/\|\nabla\tilde{v}^\varepsilon\|_{L^2(\Omega)}$ and $\alpha^\varepsilon = 1/(\|\nabla\tilde{v}^\varepsilon\|_{L^2(\Omega)}\|\varphi^\varepsilon u\|_\varepsilon)$. Therefore, the theorem is proved.

Remark 2. Note that, because of the geometry of the problem, the proof in (15.17) provides the smoothness of the eigenelements of (15.13) in Ω_+^0 and consequently in $\Omega_\pm$.

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16

On J. Ball's Fundamental Existence Theory and Regularity of Weak Equilibria in Nonlinear Radial Hyperelasticity

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In 1982, J. Ball formulated a pioneering theory on the existence and uniqueness of weak radial equilibria to the pure displacement boundary value problem associated with isotropic, frame-invariant strain-energy functions in nonlinear hyperelasticity. In the theory [Bal82], he posed the following question: "Does strong ellipticity ('of the stored energy') imply that all solutions to the equilibrium equations which pass through the origin and have *finite* energy are trivial?" J. Ball's work depended critically on the number of elasticity dimensions.

In this chapter, we will present models in n-dimensional elasticity that establish that the answer to J. Ball's question is negative. This work extends to higher dimensional elasticity the approach and results we presented, for the first time, on this question in [Ha07]. These models also provide further insight into another central, (very) difficult problem of nonlinear elasticity, namely, that of regularity of weak equilibria, which would be hard to gain by other methods such as the common, but delicate, phase plane analysis.

16.1 Introduction and Purpose

In effect, we consider a nonhomogeneous, isotropic, hyperelastic material body which occupies the open, bounded subset Ω of $\mathbb{R}^3$ in a reference configuration that we assume to be stress-free. The mechanical properties of the material body are characterized by a stored-energy density function

$$W : \Omega \times M_+^{3\times 3} \to [0, \infty).$$

To effect an extreme deformation, that is, to compress the body towards zero volume or to extend it to infinite volume requires an infinite amount of energy. This natural observation amounts to having W obey the following reasonable growth behavior for each $x \in \Omega$:

$$W(x, F) \to +\infty \quad \text{as} \quad \det F \to 0^+ \quad \text{or} \quad +\infty. \tag{16.1}$$

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We further assume W to be frame-indifferent and isotropic. That is, W respectively satisfies

$$W(x, RF) = W(x, F) \quad \text{and} \quad W(x, FR) = W(x, F) \tag{16.2}$$

for all $x \in \Omega,\ F \in M_+^{3\times 3}$, and R proper orthogonal. For details, see [TN65] and [CS94].

It is well known that (16.2) holds if and only if there exists a function $\phi : \bar{\Omega} \times (0, \infty)^3 \to \mathbb{R}$ such that, for all $F \in M_+^{3\times 3},\ \phi(x, \cdot, \cdot, \cdot)$ is symmetric and

$$W(x, F) = \phi(x, \nu_1, \nu_2, \nu_3).$$

The ν_i's are the singular values of F, and they represent the principal stretches of F. For details, see [RE55] and [TN65].

In addition to the regularity relationship between W and ϕ (e.g., $W \in C^k$ if and only if $\phi \in C^k,\ k \in \mathbb{N}$), there is an important connection between their constitutive behavior. If $W \in C^1(\Omega \times M_+^{3\times 3})$ and is rank-one convex, then ϕ satisfies the Baker–Erickson inequalities:

$$\frac{(\nu_i \Phi_i - \nu_j \Phi_j)}{(\nu_i - \nu_j)} \geq 0 \tag{16.3}$$

for $i \neq j,\ \nu_i \neq \nu_j,\ \Phi_i = \partial\phi/\partial\nu_i,\ i = 1, 2, 3$. (For further details, see [TN65].)

The Baker–Ericksen inequalities are physically a very plausible constitutive criterion inasmuch as they ensure the preservation of order between the greatest stress and the greatest extension.

In the absence of external forces, the total stored energy associated with a deformation $u(\cdot)$ of the body is given by

$$u \to J(u, \Omega) := \int_\Omega W(x, \nabla u(x))dx, \tag{16.4}$$

and the equilibrium equations are given by the Euler–Lagrange equations:

$$\operatorname{div}\left[\frac{\partial W}{\partial F}(x, F)\right] = 0, \quad x \in \Omega,$$

where $F \equiv \nabla u(x)$. For a given positive real number λ, we mainly reconsider the questions of regularity and coexistence of nontrivial equilibrium solutions of

$$\operatorname{div}\left[\frac{\partial W}{\partial F}(x, F)\right] = 0, \quad \text{in } \Omega \tag{16.5}$$

$$u(x) = \lambda x \quad \text{on} \quad \partial\Omega. \tag{16.6}$$

We hereinafter use the abbreviation (DBVP) to refer to the displacement boundary value problem consisting of equations (16.5) and (16.6). Generally, for a pure displacement boundary value problem of nonlinear elasticity, it is sufficient [AB78] to consider only those deformations for which the condition

$$\det[\nabla u(x)] > 0$$

holds for each x in Ω. (See [Mor66].) This condition ensures, among other properties, the invertibility and order preservation of the admissible deformation. We also remark that (16.6) is to be understood in the sense of the trace on $\partial\Omega$ (see [Ada75]).

By restricting the geometrical structure of Ω to be the unit ball in $\mathbb{R}^3$, Ball [Bal77b] and [Bal82] discussed favorable conditions other than rank-one convexity and quasi-convexity of the stored-energy density to ensure the uniqueness of homogeneous radial equilibria to (DBVP). In this case, the admissible deformations are considered to be of the form

$$u(x) = \frac{r(R)}{R}\, x, \tag{16.7}$$

where $R = |x|$.

For $u \in W^{1,1}(\Omega;\mathbb{R}^3)$, the weak derivatives of u in (16.7) are given by

$$\nabla u(x) = \frac{r(R)}{R}\, 1 + \frac{x \otimes x}{R^2}\left[r'(R) - \frac{r(R)}{R}\right], \quad \text{for a.e. } \; x \in \Omega. \tag{16.8}$$

Equation (16.8) implies that

$$v_1 = r', \; v_2 = v_3 = r/R.$$

The total energy functional $J(u;\Omega)$ in (16.4) now becomes $J(u;\Omega) = 4\pi I(r)$, where

$$I(r) := \int_0^1 R^2\phi(R; r', r/R, r/R)dR. \tag{16.9}$$

It is known ([Bal82], Theorem 4.2) that $u(x) = (r/R)x \in W^{1,1}(\Omega;\mathbb{R}^3)$ is a weak equilibrium solution if and only if $r'(R) > 0$ a.e. in $(0,1)$, $R^2\phi_{,1}(R)$ and $R^2\phi_{,2}(R) \in L^1(0,1)$, and $R^2\phi_{,1}(R) = 2\displaystyle\int_1^R \rho\phi_{,2}(\rho)d\rho + \text{const.}$, a.e. in $(0,1)$, where $\phi_{,i}(R) = \phi_{,i}\left(R; r', \dfrac{r(R)}{R}, \dfrac{r(R)}{R}\right)$ for $i = 1,2$. A stable equilibrium solution will minimize the functional $I(\cdot)$ of (16.9).

In [Ha07], we constructed models in plane elasticity of strongly elliptic strain-energy density functions of the form

$$W(x,F) = R^{-3}\big[\det(R(1-\gamma)F) - 1\big]^a \left[\det(R^{1-\gamma}F)\right]^{-b}, \tag{16.10}$$

where $\gamma \in (0,1)$, and a and b are positive real numbers. There we showed that, for certain choices of a and b, the equilibrium equations associated with (16.10) admit nontrivial weak solutions of the form $r(R) = \lambda R^\gamma$ for which the total energy is finite. In other words, we showed that strong ellipticity of W is not sufficient for equilibrium solutions passing through the origin and having finite energy to be trivial. In fact, those models may be modified to produce nontrivial equilibria having the same energy value as the global minimizer of the functional J, namely, zero. Nonetheless, our plan here is to generalize the approach in [Ha07] to three-dimensional elasticity. We do so in the next section, which is the main part of this work. But first we remark the following from [Ha07].

Remark 1. Let $f(R,r,r')$ denote the integrand of $I(r)$ in (16.9), namely,

$$f(R,r,r') = R^2\phi(R;r',\frac{r}{R},\ \frac{r}{R}). \tag{16.11}$$

For some $\gamma \in (0,1)$ and for every $\varepsilon > 0$ we assume that f satisfies the following constitutive property:

$$f(\varepsilon R, \varepsilon^{\gamma} r, \varepsilon^{\gamma-1} r') = \varepsilon^{-1} f(R,r,r'). \tag{16.12}$$

This homogeneity property was used in [BM85] to study the regularity of minimizers for one-dimensional variational problems in the calculus of variations. See [Hai00] for a physical interpretation of this scale-invariance property. Setting $\varepsilon = \frac{1}{R}$ in (16.12) yields

$$f(R,r,r') = R^{-1} f(1, rR^{-\gamma}, r'R^{1-\gamma}). \tag{16.13}$$

Let

$$P(R,r') = r'R^{1-\gamma} \quad \text{and} \quad X(R,r) = rR^{-\gamma}.$$

Relation (16.13) may now be rewritten as

$$f(R,r,r') := R^{-1} e(P,X), \tag{16.14}$$

where

$$e(P,X) = f(1,X,P).$$

Due to the symmetry property of $\phi(R;\cdot,\cdot,\cdot)$ in v_1, v_2, and v_3, we observe that $\phi(R;r',r/R,r/R)$ is the restriction of $\phi(R;v_1,v_2,v_3)$ to the plane $v_2 = v_3 = r/R$. Equivalently, $e(P,X)$ is the restriction to the plane $X_1 = X_2 = X$ of the symmetric quantity $E(P,X_1,X_2)$ associated with $\phi(R;v_1,v_2,v_3)$, where $X_i = v_{i+1}R^{1-\gamma}$ for $i = 1,2$. Moreover, the condition $\phi_{,11}(R;r',r/R,r/R) \geq 0$ is equivalent to $e_{,pp}(P,X) \geq 0$.

For some $\lambda \in (0,+\infty)$, observe that an $r(\cdot)$ of the form $r(R) = \lambda R^{\gamma}$ must be an absolute minimizer for $I(\cdot)$ in (16.9) because along such $r(\cdot)$ and in the light of (16.14) one has

$$I(r) = \int_0^1 R^{-1}\ e(\lambda\gamma,\lambda)dR,$$

which will yield the value zero only if there is a zero of e of the form $(\lambda\gamma,\lambda)$ in the PX-plane. So an $r(\cdot)$ of the form $r(R) = \lambda R^{\gamma}$ corresponds to a point along the line $P = \gamma X$ in the PX-plane or, equivalently, to an admissible solution of the ordinary differential equation $P = \gamma X$.

16.2 Construction of Models and Regularity of Weak Equilibria

Before we proceed with the construction of models, we would like to recapitulate the various underlying properties of the stored-energy density function ϕ in three-dimensional elasticity. In two-dimensional elasticity these properties maintained the same form. In either case, based on the foregoing analysis, ϕ or, equivalently, the integrand f in (16.11) of the total energy functional is expected to obey the following standing assumptions:

	3-D Elasticity	2-D Elasticity
(A1)	$\phi_{,11}(R;r',r/R,r/R) > 0$	$\phi_{,11}(R;r',r/R) > 0$
(A2)	$\phi(R;\nu_1,\nu_2,\nu_3)$ is symmetric in ν_1,ν_2,ν_3	$\phi(R;\nu_1,\nu_2)$ is symmetric in ν_1,ν_2
(A3)	$\lim_{\nu_1,\nu_2,\nu_3\to 0^+} \phi = \lim_{\nu_1,\nu_2,\nu_3\to +\infty} \phi = +\infty$	$\lim_{\nu_1,\nu_2\to 0^+} \phi = \lim_{\nu_1,\nu_2\to +\infty} \phi = +\infty$
(A4)	for some $\gamma \in (0,1)$ and for every $\varepsilon > 0$ $\phi(\varepsilon R;\varepsilon^{\gamma-1}r',\varepsilon^{\gamma-1}r/R,\varepsilon^{\gamma-1}r/R) = \varepsilon^{-1}\phi(R;r',r/R,r/R)$	for some $\gamma \in (0,1)$ and for every $\varepsilon > 0$ $\phi(\varepsilon R;\varepsilon^{\gamma-1}r',\varepsilon^{\gamma-1}r/R) = \varepsilon^{-1}\phi(R;r',r/R)$
(A5)	ϕ satisfies the Baker–Ericksen inequalities: $\dfrac{\nu_i\phi_{,i}-\nu_j\phi_j}{\nu_i-\nu_j} > 0$ $i \neq j \in \{1,2,3\}$	ϕ satisfies the Baker–Ericksen inequality: $\dfrac{\nu_1\phi_{,1}-\nu_2\phi_{,2}}{\nu_1-\nu_2} > 0$

Condition (A3) is the equivalent of the extreme deformation property (16.1). The natural growth condition (e.g., superlinearity in r' as $r' \to +\infty$) usually seen in existence theorems in nonlinear elastostatics could be dispensed with in the present development since the class of ϕ's which we construct enables us to obtain explicitly the energy minimizers in different spaces.

Condition (A5) represents the Baker–Ericksen inequalities which were mentioned earlier in (16.3). Condition (A4) is simply the homogeneity property (16.13) expressed in terms of ϕ. It is this property which allows us to make use of field theory and thereby enables us to obtain the desired nontrivial minimizers.

A successful model consists of a function ϕ satisfying conditions (A1)–(A5). A striking difference between the model in [Ha07] and this model is reflected by the symmetry condition (A2). In this case, $\phi(R;r',r/R,r/R)$ is the restriction to the plane $\nu_2 = \nu_3$ of the symmetric function of (A2). We now construct the model in question.

16.2.1 Construction of Models

To begin with, for a and $b \in \mathbb{R}_+$, let us consider

$$\hat{e}(P,X) := |P-X|^a (PX^2)^{-b}. \tag{16.15}$$

Since $\hat{e}$ has no zero along $P = \gamma X$, $\hat{e}$ cannot actually serve as an example by itself. However, it will be used later in the construction of our model. This $\hat{e}(P,X)$ is the restriction to the plane $X_1 = X_2 =: X$ of the symmetric function given by

$$\widehat{E}(P,X_1,X_2) := \frac{1}{2}\left[|P-X_1|^a + |P-X_2|^a + |X_1-X_2|^a\right](PX_1X_2)^{-b}. \tag{16.16}$$

Hence, condition (A2) holds for $\widehat{E}$. Let us verify that $\widehat{E}$ satisfies the rest of the conditions.

$$\widehat{E}_{,PP}(P,X) \geq 0, \quad \text{whenever} \;\; a-b-1>0,$$

$$\lim_{P,X\to+\infty \text{ or } 0^+} \hat{E} \;=\; +\infty,$$

and the Baker–Ericksen inequalities easily follow, i.e.,

$$\frac{P\widehat{E}_{,P} - X_1\widehat{E}_{,X_1}}{P-X_1}(P,X,X) = a(P-X)^{a-2}(PX^2)^{-b}[2P+X] \geq 0.$$

Thus, $\widehat{E}$ satisfies (A1)–(A5), but again it does not have a zero along $P = \lambda X_1 = \lambda X_2$. [See Remark 1 on page 164].

However, by using (16.15), we can now construct a function E with an appropriate zero. Consider the following:

$$e(P,X) := \begin{cases} 2^b(P-X)^a(PX^2)^{-b} + \varepsilon\Delta(P,X), & 0 < PX^2 \leq 2; \\ (P-X)^a(PX^2-3)^c + N(PX^2-2)^2 + \varepsilon\Delta(P,X), & 2 < PX^2 \leq 3; \\ N(PX^2-4)^2 + \varepsilon\Delta(P,X), & PX^2 > 3 \end{cases} \tag{16.17}$$

where the numbers c, N, and ε are to be determined later, and the function $\Delta(\cdot,\cdot)$ is defined by

Model 1: $\Delta(P,X) := (PX^2-4)^2$

Model 2: $\Delta(P,X) := (P-\alpha)^2(X-\alpha)^4 + (P-\gamma\alpha)^2(X-\gamma\alpha)^4,$

where $\alpha = (4/\gamma)^{1/3}$.

In either case, $\Delta(\gamma X_0, X_0) = 0 \iff X_0 = \alpha$. For instance, in Model 1, $e(P,X) = (N+\varepsilon)\Delta(P,X)$ for $PX^2 > 3$. Thus, $e(\gamma X_0, X_0) = 0$. Hence, inf $I(r)$ over $W^{1,s}(0,1)$ for $s < 1/(1-\gamma)$ is indeed zero while, for $s \geq 1/(1-\gamma)$, the curve associated with the zero of e is no longer admissible.

Note that the function e of (16.17) is the restriction to the plane $X_1 = X_2 = X$ of the symmetric function given by

$$E(P, X_1, X_2) := \begin{cases} 2^{b-1}\widehat{E}(P, X_1, X_2) + \varepsilon \Delta(P, X_1, X_2), & \text{if } 0 < PX_1X_2 \leq 2; \\ \frac{1}{2}[(P-X_1)^a + (P-X_2)^a + (X_2-X_1)^a] & (PX_1X^2-3)^c \\ \quad + N(PX_1X^2-2)^2 + \varepsilon\Delta(P, X_1, X_2), & \text{if } 2 < PX_1X_2 \leq 3; \\ N(PX_1X_2-4)^2 + \varepsilon\Delta(P, X_1, X_2), & \text{if } PX_1X_2 > 3 \end{cases} \tag{16.18}$$

where $\widehat{E}$ is as in (16.16) and Δ is given by

$$\text{Model 1:} \quad \Delta(P, X_1, X_2) := (PX_1X_2 - 4)^2$$

$$\text{Model 2:} \quad \Delta(P, X_1, X_2) := (P-\alpha)^2(X_1-\alpha)^2(X_2-\alpha)^2 + (P-\gamma\alpha)^2(X_1-\gamma\alpha)^2(X_2-\gamma\alpha)^2.$$

This establishes the symmetry of the integrand. Hence, condition (A2) is satisfied. Condition (A3) is also satisfied since it holds for $\hat{E}$ as discussed earlier and since the increment Δ is ≥ 0 everywhere.

To complete the construction there remains to ensure that E, as defined over regions II and III (see Figure 16.1), does obey the rest of the conditions, namely, (A1), (A4), and (A5).

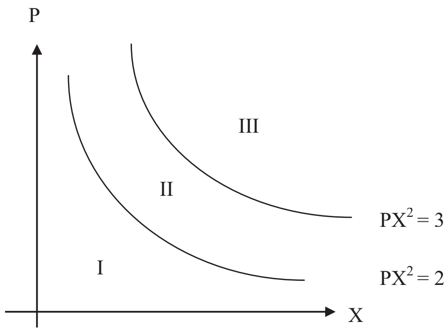

Fig. 16.1. Restriction to the plane $X_1 = X_2 = X$.

The homogeneity condition (A4) is obviously satisfied.

Since E, as defined over region III, is strictly convex in the variable PX_1X_2, then condition (A1) is immediate in that region. This also implies that E in III is rank-one convex and thus the Baker–Ericksen inequalities necessarily hold, as we discussed earlier in (16.3). Another way of verifying that condition (A5) holds for such functions is as follows.

The function E as defined over region III is of the form

$$(h \circ t)(P, X_1, X_2),$$

where $t(P, X_1, X_2) = PX_1X_2$. In terms of (P, X_1, X_2) inequalities (A5) are given by

$$\frac{P_i E_{,i} - P_j E_{,j}}{P_i - P_j} \geq 0, \qquad i \neq j = 1, 2, 3, \tag{16.19}$$

where $(P_1, P_2, P_3) = (P, X_1, X_2)$.

$$E_{,i} = h' \cdot P_j P_k, \qquad i \notin \{j, k\},$$

$$E_{,j} = h' \cdot P_k P_k, \qquad j \notin \{i, k\}.$$

$P_i E_{,i} - P_j E_{,j} \equiv 0$ and thus (A5) automatically follows. This also holds for that part of E over region II which corresponds to hydrostatic pressure (i.e., it is a function of det F). Therefore, the only less obvious part of E in region II corresponds to

$$(P - X)^a (PX^2 - 3)^c.$$

Let $g(P, X) := (P - X)^a (PX^2 - 3)^c + N(PX^2 - 2)^2$. It suffices to prove that $g_{,PP} > 0$.

Proposition 1. *We have* $g_{,PP} > 0$.

Proof. We have

$$\begin{aligned} g_{,PP} &= a(a-1)(P - X)^{a-2}(PX^2-3)^c + 2acX^2(P-X)^{a-1}(PX^2-3)^{c-1} \\ &\quad + c(c-1)X^4(P-X)^a(PX^2-3)^{c-2} + 2NX^4 \\ &= \{[a(a-1)+2ac+c(c-1)](PX^2)^2 - 6a(a-1+c)PX^2 - 2acPX^5 \\ &\quad + 6acX^3 + 2c(c-1)PX^5 + c(c-1)X^6 \\ &\quad + 9a(a-1)\}(P - zX)^{a-2}(PX^2-3)^{c-2} + 2NX^4. \end{aligned} \tag{16.20}$$

Using (16.20), we introduce $S(\cdot,\cdot)$ and $H(\cdot,\cdot)$ as follows:

$$\begin{aligned} S(P, X) &:= [a(a-1) + 2ac + c(c-1)](PX^2)^2 - 6a(a-1+c)PX^2 \\ H(P, X) &:= c(c-1)X^6 + 6acX^3 - 2acX^3(PX^2) \\ &\quad + 2c(c-1)X^3(PX^2) + 9a(a-1) + 9\ell X^4, \end{aligned}$$

where $9\ell = 2N$. We want to show that $S(P, X) \geq 0$ and $H(P, X) > 0$.

Since $PX^2 \geq 0$, then

$$\begin{aligned} S(P, X) \geq 0 &\iff a(a-1) + 2ac + c(c-1) - 3a(a-1+c) \geq 0 \\ &\iff c^2 - (1+a)c + 2a(1-a) \geq 0 \\ &\iff c \geq 2a. \end{aligned} \tag{16.21}$$

Since $2 \leq PX^2 \leq 3$, then

$$H(P,X) > 0 \Longleftrightarrow c(c-1)X^6 - 6c(c-1)X^3 + 9a(a-1) + 9\ell X^4 > 0. \quad (16.22)$$

The proof of (16.22) is a bit tricky.

Case 1. Suppose that $X \geq 1$. Then $X^4 \geq X^3$ and it is clear how to choose ℓ to make (16.22) hold. Nevertheless, let us do the following: put $X^3 =: t$,

$$H(P,X) > 0 \Longleftrightarrow Q(t) := c(c-1)t^2 - 6c(c-1)t + 9[a(a-1) + \ell] > 0.$$

Choosing ℓ such that

$$c(c-1) - a(a-1) \leq \ell,$$

implies $Q(t) > 0$ for all $t \geq 1$.

Case 2. Suppose that $X < 1$. Then $X^4 > X^6$ and

$$H(P,X) > 0 \Longleftrightarrow [c(c-1) + 9\ell]t^2 - 6c(c-1)t + 9a(a-1) =: q(t) > 0.$$

Choosing ℓ such that

$$\frac{c(c-1)}{9a(a-1)}[c(c-1) - a(a-1)] \leq \ell,$$

makes $q(t) > 0$ for all $t < 1$. Let us now take

$$\ell := \max\{c(c-1) - a(a-1),\ \frac{c(c-1)}{9a(a-1)}[c(c-1) - a(a-1)]\}. \quad (16.23)$$

With this choice of ℓ, it follows that $H(P,X) > 0$.

For $a = 2$ and $2 \leq PX^2 \leq 3$ it is not difficult to see that

$$g_{,PP} > (PX^2 - 3)^{c-2}[S + H] \qquad \text{(of course } c \geq 2a\text{)}.$$

Relations (16.21) and (16.23) imply that $g_{,PP} > 0$ (i.e., $E_{,PP}(P,X,X) > 0$ in region II.) This completes the proof of Proposition 1.

To finish the construction of the model there remains to verify that g satisfies the Baker–Ericksen inequalities. Recall that g is the restriction to the plane $X_1 = X_2 = X$ of the symmetric function given by the following expression:

$$\frac{1}{2}[(P-X_1)^a + (P-X_2)^a + (X_1-X_2)^a](PX_1X_2-3)^c + N(PX_1X_2-2)^2. \quad (16.24)$$

Clearly, the term $(PX_1X_2-2)^2$ verifies inequalities (16.19). Let $G(P, X_1X_2)$ denote the remaining term in expression (16.24). By direct computation, it follows that

$$G_{,P} = a[(P-X_1)^{a-1} + (P-X_2)^{a-1}](PX_1X_2-3)^c$$
$$+\ cX_1X_2(PX_1X_2-3)^{c-1}[(P-X_1)^a + (P-X_2)^a + (X_1-X_2)^a];$$

$$G_{,X_i} = a[-(P-X_i)^{a-1} + (X_i-X_j)^{a-1}](PX_1X_2-3)^c$$
$$+\ cPX_j(PX_1X_2-3)^{c-1}[(P-X_1)^a + (P-X_2)^a + (X_1-X_2)^a];$$

where $i \neq j \in \{1, 2\}$;

$$\frac{PG_{,P} - X_i G_{,X_i}}{P - X_i}(P, X, X) = a(P - X)^{a-2}(PX^2 - 3)^c(2P + X) \geq 0, \text{ for } i = 1, 2.$$

By symmetry the remaining inequality of (A5) also follows. Hence, E, as defined in (16.18), satisfies (A1)–(A5), and consequently the construction of the models in question is now complete.

16.3 Concluding Remarks

1. The exponent 2 in the term $\Delta(P, X)$ is immaterial (see Section 16.2.1). Choosing a sufficiently large exponent, we can therefore obtain a stored-energy function corresponding to strong materials which still exhibit a rather strong singularity at the center of the ball, just as in $r(R) = \lambda R^\gamma$.
2. There is nothing specific in choosing the boundaries of regions I and II to be $PX^2 = 2, 3$. In fact, one can take any two distinct positive real numbers k_1 and k_2, form the curves $PX^2 = k_1$ and k_2, and then closely follow the same steps as above to get similar results.
3. Our model shows that strong ellipticity of the strain energy in higher dimensional elasticity is not sufficient for equilibrium solutions passing through the origin and having finite energy to be trivial. It admits non-trivial, singular solutions of the form $r(R) = \lambda R^\gamma$ having the same energy value as the absolute minimizer!

   In n-dimensional elasticity the above model still corresponds to a natural state and also yields a nontrivial equilibrium solution, exactly the case $n = 2$ as in [Ha07]. We should also note that these solutions shape the common property r', $\frac{r}{R} \to +\infty$ as $R \to 0^+$ as do cavitation solutions in which $r(0) > 0$. This means that the singular behavior of ϕ as $v_i \to 0^+$ does not play any role in the existence of such nontrivial solutions.

This provides further insight into the fundamental question of regularity for nonconvex W's. How regular can the solution be? Can it be in $W^{1,s}(0, 1)$ for $s \geq \frac{1}{1-\gamma}$ or even in a smaller space? While very little is known about this question, it would be worthwhile investigating the possible existence of other nontrivial, singular solutions of the form $\hat{r}(R) = \lambda R^{\beta\gamma}$ for $\beta > 1$ for which the energy functional shows a gap in its infimum over the two different admissible spaces. Such singular solutions might be connected with material defectiveness such as the onset of fracture. Nonconvex strain energies (W's) are of interest since they can be connected with materials that undergo phase transitions (see [Er73]). Likewise, singular equilibrium solutions such as those above might be connected with material defectiveness such as the onset of fracture. These issues also shed light on another equally important question: that of obtaining formulations of the problems that are amenable to successful numerical treatments.

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17

The Conformal Mapping Method for the Helmholtz Equation

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The Helmholtz equation describes a lot of physical processes. For example, in quantum chaos some model systems are described by the Helmholtz equation with appropriate boundary conditions. One of them is the quantum billiard problem (see [Bu01], [Gr01], [Gu90], [KoSc97], [Si00], and [Si70]).

Generic billiards are one of the simplest examples of conservative dynamical systems with chaotic classical trajectories.

According to this model, the particle is trapped inside the simply corrected region D with the boundary S, in which it can move freely and this movement is ballistic.

In this case, the Schrödinger equation for a free particle assumes the form of the Helmholtz equation (see [Gr01], [Gu90], [Si00], and [Si70]).

This chapter deals with the two-dimensional homogeneous problem for the Helmholtz equation in the finite domain D with the boundary S. The following problem is considered.

Problem 1. Find a real function $u(x, y)$ in D having second-order derivatives satisfying the equation

$$\Delta u(x, y) + \lambda^2 u(x, y) = 0$$

and the boundary condition

$$u\big|_S = 0,$$

where λ is a constant to be determined.

The constant λ^2 reflects the energy levels of the particle.

We need to calculate the eigenvalues and eigenfunctions for the Dirichlet boundary conditions (hard-wall conditions) of Problem 1.

The spectrum of this equation is discrete, and the distribution of the energy levels is determined by the form of the area [Bi88].

In this chapter, Problem 1 is investigated by means of the conformal mapping and integral equation methods, and particular cases are considered (hexagon, cardioid, and lemniscate).

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Let $z = f(\zeta)$ be the conformal mapping of the rectangle $D_0\{0 \le \xi \le a;$ $0 \le \eta \le b\}$ of $\zeta, (\zeta = \xi + i\eta)$, plane on the area D of the complex z-plane where $(z = x + iy)$. The boundary of D_0 is denoted by S_0.

This mapping reduces Problem 1 to the following problem.

Problem 2. Find a real function $u_0(x, y)$ in D_0 having second-order derivatives and satisfying the equation and boundary condition

$$\Delta u_0(\xi,\eta) + \lambda^2 \left|f'(\zeta)\right|^2 u_0(\xi,\eta) = 0, \qquad (17.1)$$
$$u_0\big|_{S_0} = 0,$$

where $u_0(\xi, \eta) = u(f(\zeta))$ and λ is a constant to be determined.

Using the Poisson representation, we can reduce Problem 2 to the equivalent integral equation [Bi88]

$$u_0(\zeta_0) - \frac{\lambda^2}{2\pi} \int_{D_0} \left|f'(\zeta)\right|^2 K(\zeta,\zeta_0)\, u_0(\zeta)\, d\xi\, d\eta = 0, \quad \zeta_0 \in D_0, \qquad (17.2)$$
$$\zeta_0 = \xi_0 + i\eta_0,$$

where

$$K(\zeta,\zeta_0) = \ln \left| \frac{\sigma(\zeta - \zeta_0)\, \sigma(\zeta + \zeta_0)}{\sigma(\zeta - \overline{\zeta}_0)\, \sigma(\zeta + \overline{\zeta}_0)} \right|.$$

Here is a definite branch of this function, σ is the Weierstrass function for the periods $2a$ and $2b$, $\overline{\zeta}_0 = \xi_0 - i\eta_0$ (see [LaSh87] and [JaEnLo60]), and σ is given by the formulas

$$\sigma(\zeta) = \frac{2ae^{\delta\zeta^2/4a}}{\theta_1'(0)}\, \theta_1\Big(\frac{\zeta}{2a}\Big);$$

$$\ln \theta_1(\zeta) = \ln \sin \pi\zeta + \sum_{n=1}^{\infty} \frac{q^n \cos 2\pi n\zeta}{n \sinh \pi n \varkappa}; \quad q = e^{-\pi\varkappa}; \quad \varkappa = \frac{b}{a}, \qquad (17.3)$$

where θ_1 is the Jacobi function and δ is a specific constant.

Using Banach's theorem, we easily prove the next assertion.

Theorem 1. *If*

$$\frac{\lambda^2}{2\pi} < \frac{1}{d(D)}\,,$$

where $d(D)$ is the diameter of D, then equation (17.2) has only the trivial solution.

Let us introduce the notation $v(\zeta) = |f'(\zeta)|^2 u_0$; then we can rewrite equation (17.2) in the form

$$v(\zeta_0) - \frac{\lambda^2}{2\pi}\big|f'(\zeta)\big|^2 \int_{D_0} K(\zeta,\zeta_0)\, v(\zeta)\, d\xi\, d\eta = 0, \quad \zeta_0 \in D_0, \tag{17.4}$$

$$\zeta_0 = \xi_0 + i\eta_0.$$

We admit that the function u in the rectangle D_0 is representable by the Fourier series

$$u = \sum_{m,n} c_{mn} \sin \frac{m\pi}{a}\, \xi \, \sin \frac{n\pi}{b}\, \eta. \tag{17.5}$$

Substituting (17.5) into (17.4), we obtain

$$v(\zeta_0) - \frac{\lambda^2}{2\pi}\, |f'(\zeta_0)|^2 \sum_{m,n} C_{mn} \int_{D_0} K(\zeta,\zeta_0)\, \sin \frac{m\pi}{a}\, \xi\, \sin \frac{n\pi}{b}\, \eta\, d\xi\, d\eta = 0. \tag{17.6}$$

Multiplying (17.6) by $\sqrt{\frac{4}{ab}}\, \sin m_1 \frac{\pi}{a}\, \xi_0\, \sin \frac{n_1\pi}{b}\, \eta_0$ and integrating over the rectangle D_0, we obtain

$$C_{m_1n_1} - \frac{\lambda^2}{2\pi} \sum_{m,n} C_{mn}\, f^{mn}_{m_1n_1} = 0, \quad m_1, n_1 = 1, 2, \ldots, \tag{17.7}$$

where

$$f^{mn}_{m_1n_1} = \sqrt{\frac{4}{ab}} \int_{D_0}\int_{D_0} \big|f'(\zeta)\big|^2 \sin \frac{m\pi}{a}\, \xi\, \sin \frac{n\pi}{a}\, \eta$$
$$\times \sin \frac{m_1\pi}{a} \xi_0\, \sin \frac{n_1\pi}{a}\, \eta_0\, d\xi\, d\eta d\xi_0\, d\eta_0.$$

The formula (17.7) represents the infinite system of homogeneous linear algebraic equations with respect to $C_{m_1n_1}$.

As the three parameters of the conformal mapping can be chosen arbitrarily, we can assume, for example, that $\varkappa = \frac{b}{a} = 10$, $(a = 1,\ b = 10)$, so $q = e^{-\pi\varkappa}$ (see (17.3)) will be sufficiently small and the series in (17.7) is convergent; then with a high accuracy we can write the approximate formula

$$C_{m_1n_1} - \frac{\lambda^2}{2\pi} \sum_{m,n}^{m_0,m_0} C_{mn}\, f^{mn}_{m_1n_1} = 0, \quad m_1, n_1 = 1, \ldots, m_0. \tag{17.8}$$

This is a finite system of homogeneous linear algebraic equations. As we seek a nonzero solution, the matrix of this system should be singular; on the diagonal of this matrix we will have the terms $1 - \frac{\lambda^2}{2\pi}\, f^{11}_{11}, 1 - \frac{\lambda^2}{2\pi}\, f^{12}_{12}, \ldots$.

The determinant of this system should be zero, and we obtain an equation of the $(m_0)^2$-th order with respect to λ^2.

Let us consider some examples.

For polygonal areas, $|f'(\xi)|$ takes the form

$$f'(\zeta) = C \prod_{j=1}^{n} \left(C_0 \text{ sn } \zeta - a_j\right)^{a_j - 1} \text{ cn } \zeta \cdot \text{ dn } \zeta, \tag{17.9}$$

where C and C_0 are specific constants, $a_1, \ldots, a_n$ are the points corresponding to the vertices of the polygon, and $c_1, \ldots, c_n$ and $a_j\pi$ are the angles of D, $j = 1, \ldots, n$.

For a hexagon, (17.9) becomes

$$f'(\zeta) = C\Big\{C_0 \text{ sn } \zeta\big(c_0^2 \text{ sn }^2\zeta - a_1^2\big)\big(c_0^2 \text{ sn }^2\zeta - a_2^2\big)\Big\}^{-\frac{1}{3}} \text{ cn } \zeta \,\text{dn } \zeta.$$

We can use the formulas for the small q [JaEnLo60],

$$\begin{aligned}
\text{sn } u &\approx \sin \frac{\pi n}{2a}\left(1 + 4q\cos^2 \frac{\pi n}{2a}\right),\\
\text{cn } u &\approx \cos \frac{\pi n}{2a}\left(1 - 4q\sin^2 \frac{\pi n}{2a}\right),\\
\text{dn } u &\approx 1 - 8q\sin^2 \frac{\pi n}{2a},
\end{aligned}$$

and $C = \frac{3|d|}{2\pi}$ and $C_0 \approx 3.2$. After simple transformations we can calculate the coefficients $f_{m_1 n_1}^{nn}$ using Mathcad or Maple, so we can find the eigenvalues of system (17.8), and consequently, the corresponding independent solutions C_{mn} (Fourier coefficients of u). Thus, we obtain approximate solutions of Problem 1.

Remark 1. In some cases, it is more convenient to use mapping on the circle. Thus,

(i) for the cardioid,

$$z = f(\zeta) = \sqrt{\zeta}, \quad f'(\zeta) = \frac{1}{2\sqrt{\zeta}};$$

(ii) for the lemniscate,

$$z = f(\zeta) = (\zeta)^2, \quad f'(\zeta) = 2\zeta.$$

In these cases, is not necessary to consider the integral equation. We consider equation (17.1) and, by separation of variables, obtain the solutions of Problem 1 directly in terms of the Hankel functions.

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18

Integral Equation Method in a Problem on Acoustic Scattering by a Thin Cylindrical Screen with Dirichlet and Impedance Boundary Conditions on Opposite Sides of the Screen

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18.1 Introduction

A problem on scattering acoustic waves by a thin cylindrical screen is studied. In doing so, the Dirichlet condition is specified on one side of the screen, while the impedance boundary condition is specified on the other side of the screen. The solution of the problem is subject to the radiation condition at infinity and to the propagative Helmholtz equation. By using potential theory, the scattering problem is reduced to a system of singular integral equations with additional conditions. By regularization and subsequent transformations, this system is reduced to a vector Fredholm equation of the second kind and index zero. It is proved that the obtained vector Fredholm equation is uniquely solvable. Therefore, the integral representation for a solution of the original scattering problem is obtained.

18.2 Statement and Solution of the Problem

Consider a simple open arc $\Gamma \in C^{2,\lambda}$, $\lambda \in (0,1]$, in a plane $x \in \mathbb{R}^2$. The arc Γ is parameterized by the arc length s: $\Gamma = \{x\colon\ x = x(s) = (x_1(s), x_2(s)),\ s \in [a,b]\}$ so that $a < b$. Let τ_x and $\mathbf{n}_x$ be a tangent and a normal vector to Γ at the point $x(s)$. We consider Γ as a cut. We denote by Γ^+ the side of Γ which is to the left when the parameter s increases, and by Γ^- the opposite side.

Function $u(x)$ belongs to the smoothness class $\mathbf{K}$ if

1) $u \in C^0\left(\overline{\mathbb{R}^2 \setminus \Gamma}\right) \cap C^2\left(\mathbb{R}^2 \setminus \Gamma\right)$ and $u(x)$ is continuous at the endpoints of the arc Γ;

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2) $\nabla u \in C^0\left(\overline{\mathbb{R}^2 \setminus \Gamma} \setminus X\right)$, where X is the set of endpoints of Γ: $X = x(a) \cup x(b)$;
3) in a neighborhood of each endpoint $x(d) \in X$ for some constants $\mathcal{C} > 0$, $\epsilon > -1$, the inequality $|\nabla u| \leq \mathcal{C}|x - x(d)|^{\epsilon}$ holds when $x \to x(d)$ and $d = a$ or $d = b$.

Here functions $u(x)$ and $\nabla u(x)$ are continuously extendable at the cut $\Gamma \setminus X$ from the left and from the right, but they may have a jump across $\Gamma \setminus X$.

The problem. Find a function $u(x)$ of the class $\mathbf{K}$, which satisfies the Helmholtz equation

$$\Delta u(x) + k^2 u(x) = 0, \quad x \in \mathbb{R}^2 \setminus \Gamma, \quad k = \text{const} > 0, \tag{18.1}$$

the boundary conditions

$$u(x)|_{x(s)\in\Gamma^+} = f^+(s), \tag{18.2}$$

$$\left[\frac{\partial u(x)}{\partial \mathbf{n}_x} - \beta(s)u(x)\right]\Bigg|_{x(s)\in\Gamma^-} = f(s), \tag{18.3}$$

and the Sommerfeld radiation condition at infinity

$$u = O\left(|x|^{-1/2}\right), \quad \frac{\partial u(x)}{\partial |x|} - iku(x) = o\left(|x|^{-1/2}\right), \tag{18.4}$$

where $|x| = \sqrt{x_1^2 + x_2^2} \to \infty$. Here $\beta(s)$, $f(s) \in C^{0,\lambda}(\Gamma)$, $f^+(s) \in C^{1,\lambda}(\Gamma)$, and $\mathrm{Im}\beta(s) \leq 0$ for each $s \in \Gamma$.

Taking into account (18.2), condition (18.3) can be replaced by the equivalent condition

$$\frac{\partial u(x)}{\partial \mathbf{n}_x}\Bigg|_{x(s)\in\Gamma^-} + \beta(s)\left[u(x)|_{x(s)\in\Gamma^+} - u(x)|_{x(s)\in\Gamma^-}\right] = f^-(s), \tag{18.5}$$

where $f^-(s) = f(s) + \beta(s)f^+(s) \in C^{0,\lambda}(\Gamma)$.

Boundary condition (18.2) can be differentiated in terms of s, and we obtain the conditions

$$\frac{\partial u(x)}{\partial \tau_x}\Bigg|_{x(s)\in\Gamma^+} = \left(f^+\right)'(s), \tag{18.6}$$

$$u\big(x(a)\big) = f^+(a). \tag{18.7}$$

We can prove that the problem has no more then one solution. We shall look for a solution to the problem (18.1)–(18.4) of the form

$$u[\mu,\nu](x) = T[\mu](x) + W[\nu](x), \tag{18.8}$$

where

$$T[\mu](x) = \frac{i}{4} \int_{\Gamma} \mu(\sigma) V(x,\sigma)\, d\sigma$$

is the angular potential introduced by Gabov,

$$V(x,\sigma) = \int_a^{\sigma} \frac{\partial}{\partial \mathbf{n}_y} \mathcal{H}_0^{(1)}\big(k|x-y(\xi)|\big)\, d\xi, \quad \sigma \in [a,b],$$

$$W[\nu](x) = \frac{i}{4} \int_{\Gamma} \nu(\sigma) \mathcal{H}_0^{(1)}\big(k|x-y(\sigma)|\big)\, d\sigma$$

is the single-layer potential, and $\mathcal{H}_0^{(1)}(z)$ is the Hankel function of the first kind and index zero.

Densities $\mu(s)$, $\nu(s)$ in potentials are of space $C_q^{\omega}(\Gamma)$, $\omega \in (0,1]$, $q \in [0,1)$. We say that $\mathcal{F}(s) \in C_q^{\omega}(\Gamma)$, if $\mathcal{F}_0(s) \in C^{0,\omega}(\Gamma)$, where $\mathcal{F}_0(s) = \mathcal{F}(s)(s-a)^q$ $(b-s)^q$, and $\|\mathcal{F}(\cdot)\|_{C_q^{\omega}(\Gamma)} = \|\mathcal{F}_0(\cdot)\|_{C^{0,\omega}(\Gamma)}$.

Furthermore, function $\mu(s)$ must satisfy the condition

$$\int_a^b \mu(\sigma)\, d\sigma = 0. \tag{18.9}$$

Using [Kr94(1)], we can prove that function (18.8) fulfills all the conditions of the problem except the boundary conditions. We substitute (18.8) in (18.5), and (18.6) and obtain the integral equations for $\mu(s)$ and $\nu(s)$ on Γ:

$$\mu(s) + \frac{1}{\pi} \int_{\Gamma} \nu(\sigma) \frac{d\sigma}{\sigma - s} + \int_{\Gamma} \mu(\sigma) w_1(s,\sigma)\, d\sigma + \int_{\Gamma} \nu(\sigma) w_2(s,\sigma)\, d\sigma = 2\left(f^+\right)'(s), \tag{18.10}$$

$$-\nu(s) - \frac{1}{\pi} \int_{\Gamma} \mu(\sigma) \frac{d\sigma}{\sigma - s} + \int_{\Gamma} \nu(\sigma) w_3(s,\sigma)\, d\sigma - \int_{\Gamma} \mu(\sigma) w_4(s,\sigma)\, d\sigma + 2\beta(s)\rho[\mu](s) = 2f^-(s), \tag{18.11}$$

where

$$w_1(s,\sigma) = \frac{1}{\pi} \frac{\cos\varphi_0\big(x(s),y(\sigma)\big)}{|x(s)-y(\sigma)|} + \frac{i}{2}\frac{\partial}{\partial s} V_0\big(x(s),\sigma\big),$$

$$w_2(s,\sigma) = \frac{1}{\pi}\left(\frac{\sin\varphi_0\big(x(s),y(\sigma)\big)}{|x(s)-y(\sigma)|} - \frac{1}{\sigma - s}\right) + \frac{i}{2}\frac{\partial}{\partial s} h\big(|x(s)-y(\sigma)|\big),$$

$$w_3(s,\sigma) = \frac{1}{\pi} \frac{\cos\varphi_0\big(x(s),y(\sigma)\big)}{|x(s)-y(\sigma)|} + \frac{i}{2}\frac{\partial}{\partial \mathbf{n}_x} h\big(|x(s)-y(\sigma)|\big),$$

$$w_4(s,\sigma) = \frac{1}{\pi}\left(\frac{\sin\varphi_0\big(x(s),y(\sigma)\big)}{|x(s)-y(\sigma)|} - \frac{1}{\sigma - s}\right) - \frac{i}{2}\frac{\partial}{\partial \mathbf{n}_x} V_0\big(x(s),\sigma\big),$$

$$V_0(x,\sigma) = \int_a^\sigma \frac{\partial}{\partial \mathbf{n}_y} h\big(k|x-y(\xi)|\big)\,d\xi,$$

$$\rho[\mu](s) = \int_a^s \mu(\sigma)\,d\sigma, \quad s \in [a,b],$$

$$h(z) = \mathcal{H}_0^{(1)}(z) - \frac{2i}{\pi}\ln\frac{z}{k}.$$

By $\varphi_0(x,y)$ we have denoted the angle between $\overrightarrow{xy}$ and $\mathbf{n}_x$ measured anticlockwise.

According to [Kr94(1), Kr94(2)], $w_3(s,\sigma) \in C^{0,\lambda}(\Gamma \times \Gamma)$ and $w_j(s,\sigma) \in C^{0,p_0}(\Gamma \times \Gamma)$ when j =1, 2, 4. Here $p_0 = \lambda$, if $0 < \lambda < 1$, and $p_0 = 1 - \varepsilon_0$ for each $\varepsilon_0 \in (0,1)$, if $\lambda = 1$.

Substituting function (18.8) in condition (18.7), we obtain one more equation for $\mu(s)$, $\nu(s)$:

$$T[\mu]\big(x(a)\big) + W[\nu]\big(x(a)\big) = 0. \tag{18.12}$$

Then we make the change of unknown densities $\mu(s)$, $\nu(s)$, so that the characteristic part of singular integral equations (18.10), (18.11) contains only one unknown function. After regularization of these equations, using (18.9) and (18.12), we obtain a vector Fredholm equation of index zero. The homogeneous equation has only a trivial solution. It means that the nonhomogeneous equation is uniquely solvable. So, system (18.9)–(18.12) is uniquely solvable.

Theorem 1. *Problem (18.1)–(18.4) has a unique solution, given by the sum of potentials (18.8) with densities satisfying uniquely solvable Fredholm equations of the second kind and index zero.*

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Existence of a Classical Solution and Nonexistence of a Weak Solution to the Dirichlet Problem for the Laplace Equation in a Plane Domain with Cracks

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19.1 Introduction

Plane domains with cracks are plane domains bounded by closed curves and open arcs (cracks). Boundary value problems in such domains model cracked solid bodies or obstacles and screens (or wings) in fluids. An integral representation of a classical solution to the harmonic Dirichlet problem in a plane domain with cracks of an arbitrary shape has been obtained by the method of integral equations in [Kr00-1], [Kr00-2], [Kr98], [Kr97], [Kr05] in the case when the solution is assumed to be continuous at the ends of the cracks. In this chapter this problem is considered in the case when the solution is not continuous at the ends of the cracks. The well-posed formulation of the boundary value problem is given, theorems on existence and uniqueness of a classical solution are proved, and the integral representation for a classical solution is obtained. Moreover, properties of the solution are studied with the help of this integral representation. It appears that the classical solution to the Dirichlet problem considered in this chapter exists, while the weak solution typically does not exist, though both the cracks and the functions specified in the boundary conditions are smooth enough. This result follows from the fact that the square of the gradient of a classical solution basically is not integrable near the ends of the cracks, since singularities of the gradient are rather strong there. This result is very important for numerical analysis; it shows that finite elements and finite difference methods cannot be applied to numerical treatment of the Dirichlet problem in question directly, since all these methods imply existence of a weak solution. To use difference methods for numerical analysis, one has to localize all strong singularities first and next use a difference method in a domain excluding the neighborhoods of the singularities.

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19.2 Formulation of the Problem

By an open curve we mean a simple smooth nonclosed arc of finite length without self-intersections [Mu68].

In a plane in Cartesian coordinates $x = (x_1, x_2) \in R^2$ we consider a connected domain bounded by simple open curves $\Gamma_1^1, \ldots, \Gamma_{N_1}^1 \in C^{2,\lambda}$ and simple closed curves $\Gamma_1^2, \ldots, \Gamma_{N_2}^2 \in C^{2,\lambda}$, $\lambda \in (0,1]$, in such a way that all curves are disjoint. We will consider both the case of an exterior domain and the case of an interior domain when the curve Γ_1^2 encloses all others. Set

$$\Gamma^1 = \bigcup_{n=1}^{N_1} \Gamma_n^1, \qquad \Gamma^2 = \bigcup_{n=1}^{N_2} \Gamma_n^2, \qquad \Gamma = \Gamma^1 \cup \Gamma^2.$$

The connected domain bounded by closed curves Γ^2 and containing open curves Γ^1 will be called $\mathcal{D}$, so that $\partial\mathcal{D} = \Gamma^2$, $\Gamma^1 \subset \mathcal{D}$. We assume that each curve Γ_n^j is parametrized by the arc length s:

$$\Gamma_n^j = \left\{x : \quad x = x(s) = \Big(x_1(s), x_2(s)\Big), \quad s \in \left[a_n^j, b_n^j\right]\right\},$$

$$n = 1, \ldots, N_j, \quad j = 1, 2,$$

so that $a_1^1 < b_1^1 < \cdots < a_{N_1}^1 < b_{N_1}^1 < a_1^2 < b_1^2 < \cdots < a_{N_2}^2 < b_{N_2}^2$ and the domain $\mathcal{D}$ is placed to the right when the parameter s increases on Γ_n^2. The points $x \in \Gamma$ and values of the parameter s are in one-to-one correspondence except for the points a_n^2, b_n^2, which correspond to the same point x for $n = 1, \ldots, N_2$. Further on, the set of the intervals $\bigcup_{n=1}^{N_1} \left[a_n^1, b_n^1\right]$, $\bigcup_{n=1}^{N_2} \left[a_n^2, b_n^2\right]$, $\bigcup_{j=1}^{2} \bigcup_{n=1}^{N_j} \left[a_n^j, b_n^j\right]$ on the Os-axis will be denoted by Γ^1, Γ^2, and Γ also.

Set $C^{j,r}\left(\Gamma_n^2\right) = \Big\{\mathcal{F}(s)\colon \ \mathcal{F}(s) \in C^{j,r}\left[a_n^2, b_n^2\right], \ \mathcal{F}^{(m)}\left(a_n^2\right) = \mathcal{F}^{(m)}\left(b_n^2\right),$ $m = 0, \ldots, j\Big\}$, $j = 0, 1$, $r \in [0,1]$, and $C^{j,r}\left(\Gamma^2\right) = \bigcap_{n=1}^{N_2} C^{j,r}\left(\Gamma_n^2\right)$. The tangent vector to Γ in the point $x(s)$, in the direction of the increment of s, will be denoted by $\tau_x = (\cos\alpha(s), \sin\alpha(s))$, while the normal vector coinciding with τ_x after rotation through an angle of $\pi/2$ in the counterclockwise direction will be denoted by $\mathbf{n}_x = (\sin\alpha(s), -\cos\alpha(s))$. According to the chosen parametrization, $\cos\alpha(s) = x_1'(s)$, $\sin\alpha(s) = x_2'(s)$. Thus, $\mathbf{n}_x$ is an interior normal to $\mathcal{D}$ on Γ^2. By X we denote the point set consisting of the endpoints of Γ^1: $X = \bigcup_{n=1}^{N_1} \Big(x\left(a_n^1\right) \cup x\left(b_n^1\right)\Big)$.

Let the plane be cut along Γ^1. We consider Γ^1 as a set of cracks (or cuts). The side of the crack Γ^1, which is situated on the left when the parameter s

increases, will be denoted by $\left(\Gamma^1\right)^+$, while the opposite side will be denoted by $\left(\Gamma^1\right)^-$.

We say that the function $u(x)$ belongs to the smoothness class $\mathbf{K}_1$ if

1. $u \in C^0\left(\overline{\mathcal{D} \setminus \Gamma^1} \setminus X\right) \cap C^2\left(\mathcal{D} \setminus \Gamma^1\right), \quad \nabla u \in C^0\left(\overline{\mathcal{D} \setminus \Gamma^1} \setminus \Gamma^2 \setminus X\right),$
2. in the neighborhood of any point $x(d) \in X$ the equality

$$\lim_{r \to +0} \int_{\partial S(d,r)} u(x) \frac{\partial u(x)}{\partial \mathbf{n}_x} dl = 0 \tag{19.1}$$

holds, where the curvilinear integral of the first kind is taken over a circumference $\partial S(d,r)$ of a radius r with the center in the point $x(d)$, $\mathbf{n}_x$ is a normal in the point $x \in \partial S(d,r)$, directed to the center of the circumference, and $d = a_n^1$ or $d = b_n^1$, $n = 1, \ldots, N_1$.

Remark 1. By $C^0\left(\overline{\mathcal{D} \setminus \Gamma^1} \setminus X\right)$ we denote the class of continuous in $\overline{\mathcal{D}} \setminus \Gamma^1$ functions, which are continuously extensible to the sides of the cracks $\Gamma^1 \setminus X$ from the left and from the right, but their limiting values on $\Gamma^1 \setminus X$ can be different from the left and from the right, so that these functions may have a jump on $\Gamma^1 \setminus X$. To obtain the definition of the class $C^0\left(\overline{\mathcal{D} \setminus \Gamma^1} \setminus \Gamma^2 \setminus X\right)$ we have to replace $C^0\left(\overline{\mathcal{D} \setminus \Gamma^1} \setminus X\right)$ by $C^0\left(\overline{\mathcal{D} \setminus \Gamma^1} \setminus \Gamma^2 \setminus X\right)$ and $\overline{\mathcal{D}} \setminus \Gamma^1$ by $\mathcal{D} \setminus \Gamma^1$ in the previous sentence.

Problem $\mathbf{D}_1$. Find a function $u(x)$ from the class $\mathbf{K}_1$, so that $u(x)$ obeys the Laplace equation

$$u_{x_1x_1}(x) + u_{x_2x_2}(x) = 0, \tag{19.2}$$

in $\mathcal{D} \setminus \Gamma^1$ and satisfies the boundary conditions

$$u(x)|_{x(s) \in (\Gamma^1)^+} = F^+(s), \quad u(x)|_{x(s) \in (\Gamma^1)^-} = F^-(s), \quad u(x)|_{x(s) \in \Gamma^2} = F(s). \tag{19.3}$$

If $\mathcal{D}$ is an exterior domain, then we add the following condition at infinity:

$$|u(x)| \le const, \qquad |x| = \sqrt{x_1^2 + x_2^2} \to \infty. \tag{19.4}$$

All conditions of the problem $\mathbf{D}_1$ must be satisfied in a classical sense. The boundary conditions (19.3) on Γ^1 must be satisfied in the interior points of Γ^1; their validity at the ends of Γ^1 is not required.

Theorem 1. *If $\Gamma \in C^{2,\lambda}$, $\lambda \in (0,1]$, then there is no more than one solution to the problem $\mathbf{D}_1$.*

It is enough to prove that the homogeneous problem $\mathbf{D}_1$ admits the trivial solution only. The proof will be given for an interior domain $\mathcal{D}$. Let $u^0(x)$ be a solution to the homogeneous problem $\mathbf{D}_1$ with $F^+(s) \equiv F^-(s) \equiv 0$, $F(s) \equiv 0$.

Let $S(d,\epsilon)$ be a disc of a small enough radius ϵ with the center in the point $x(d)$ $(d=a_n^1$ or $d=b_n^1,\ \ n=1,...,N_1)$. Let $\Gamma_{n,\epsilon}^1$ be a set consisting of such points of the curve Γ_n^1 which do not belong to discs $S(a_n^1,\epsilon)$ and $S(b_n^1,\epsilon)$. We choose a number ϵ_0 small enough so that the following conditions are satisfied:
1) for any $0<\epsilon\le\epsilon_0$ the set of points $\Gamma_{n,\epsilon}^1$ is a unique nonclosed arc for each $n=1,...,N_1$,
2) the points belonging to $\Gamma\setminus\Gamma_n^1$ are placed outside the discs $S(a_n^1,\epsilon_0)$, $S(b_n^1,\epsilon_0)$ for any $n=1,...,N_1$,
3) discs of radius ϵ_0 with centers in different ends of Γ^1 do not intersect.
Set $\Gamma^{1,\epsilon}=\cup_{n=1}^{N_1}\Gamma_{n,\epsilon}^1$, $S_\epsilon=\left(\cup_{n=1}^{N_1}[S(a_n^1,\epsilon)\cup S(b_n^1,\epsilon)]\right)$, $\mathcal{D}_\epsilon=\mathcal{D}\setminus\Gamma^{1,\epsilon}\setminus S_\epsilon$. Since $\Gamma^2\in C^{2,\lambda}$, $u^0(x)\in C^0(\overline{\mathcal{D}}\setminus\Gamma^1)$ (recall that $u^0(x)\in\mathbf{K}_1$), and since $u^0|_{\Gamma^2}=0\in C^{2,\lambda}(\Gamma^2)$, and owing to the theorem on regularity of solutions of elliptic equations near the boundary [GiTr77], we obtain $u^0(x)\in C^1(\overline{\mathcal{D}}\setminus\Gamma^1)$. Since $u^0(x)\in\mathbf{K}_1$, we observe that $u^0(x)\in C^1(\overline{\mathcal{D}}_\epsilon)$ for any $\epsilon\in(0,\epsilon_0]$. By $C^1(\overline{\mathcal{D}}_\epsilon)$ we mean $C^1(\mathcal{D}_\epsilon\cup\Gamma^2\cup(\Gamma^{1,\epsilon})^+\cup(\Gamma^{1,\epsilon})^-\cup\partial S_\epsilon)$. Since the boundary of a domain $\mathcal{D}_\epsilon$ is piecewise smooth, we write out Green's formula [Vl81, p. 328] for the function $u^0(x)$:

$$\|\nabla u^0\|_{L_2(\mathcal{D}_\epsilon)}^2=\int_{\Gamma^{1,\epsilon}}(u^0)^+\left(\frac{\partial u^0}{\partial\mathbf{n}_x}\right)^+ds$$

$$-\int_{\Gamma^{1,\epsilon}}(u^0)^-\left(\frac{\partial u^0}{\partial\mathbf{n}_x}\right)^-ds-\int_{\Gamma^2}u^0\frac{\partial u^0}{\partial\mathbf{n}_x}ds+\int_{\partial S_\epsilon}u^0\frac{\partial u^0}{\partial\mathbf{n}_x}dl.$$

We denote by $\mathbf{n}_x$ the exterior (with respect to $\mathcal{D}_\epsilon$) normal on ∂S_ϵ at the point $x\in\partial S_\epsilon$. By the superscripts $+$ and $-$ we denote the limiting values of functions on $(\Gamma^1)^+$ and on $(\Gamma^1)^-$, respectively. Since $u^0(x)$ satisfies the homogeneous boundary conditions (19.3) on Γ, we observe that $u^0|_{\Gamma^2}=0$ and $(u^0)^\pm|_{\Gamma^{1,\epsilon}}=0$ for any $\epsilon\in(0,\epsilon_0]$. Therefore,

$$\|\nabla u^0\|_{L_2(\mathcal{D}_\epsilon)}^2=\int_{\partial S_\epsilon}u^0\frac{\partial u^0}{\partial\mathbf{n}_x}dl,\qquad\epsilon\in(0,\epsilon_0].$$

Setting $\epsilon\to+0$, taking into account that $u^0(x)\in\mathbf{K}_1$, and using the relationship (19.1), we obtain $\|\nabla u^0\|_{L_2(\mathcal{D}\setminus\Gamma^1)}^2=\lim_{\epsilon\to+0}\|\nabla u^0\|_{L_2(\mathcal{D}_\epsilon)}^2=0$. From the homogeneous boundary conditions (19.3) we conclude that $u^0(x)\equiv0$ in $\mathcal{D}\setminus\Gamma^1$, where $\mathcal{D}$ is an interior domain. If $\mathcal{D}$ is an exterior domain, then the proof is analogous, but we have to use the condition (19.4) and the theorem on behavior of a gradient of a harmonic function at infinity [Vl81, p. 373]. The maximum principle cannot be used for the proof of the theorem even in the case of an interior domain $\mathcal{D}$, since the solution to the problem may not satisfy the boundary conditions (19.3) at the ends of the cracks, and it may not be continuous at the ends of the cracks.

19.3 Existence of a Classical Solution

Let us turn to solving the problem $\mathbf{D}_1$. Consider the double-layer harmonic potential with the density $\mu(s)$ specified at the open arcs Γ^1:

$$w[\mu](x) = -\frac{1}{2\pi}\int_{\Gamma^1}\mu(\sigma)\frac{\partial}{\partial \mathbf{n}_y}\ln|x-y(\sigma)|d\sigma. \tag{19.5}$$

Theorem 2. *Let $\Gamma^1 \in C^{1,\lambda},\ \lambda \in (0,1]$. Let $S(d,\epsilon)$ be a disc of a small enough radius ϵ with the center in the point $x(d)$ $(d = a_n^1$ or $d = b_n^1,\ n = 1,...,N_1)$.*
I. If $\mu(s) \in C^{0,\lambda}(\Gamma^1)$, then $w[\mu](x) \in C^0(\overline{R^2 \setminus \Gamma^1} \setminus X)$ and for any $x \in S(d,\epsilon)$ such that $x \notin \Gamma^1$ the inequality holds

$$|w[\mu](x)| \le \text{const}.$$

II. If $\mu(s) \in C^{1,\lambda}(\Gamma^1)$, then
1) $\nabla w[\mu](x) \in C^0(\overline{R^2 \setminus \Gamma^1} \setminus X)$;
2) for any $x \in S(d,\epsilon)$ such that $x \notin \Gamma^1$, the following formulas hold:

$$\frac{\partial w[\mu](x)}{\partial x_1} = \frac{1}{2\pi}\frac{\mp\mu(d)}{|x-x(d)|}\sin\psi(x,x(d)) + \Omega_1(x),$$

$$\frac{\partial w[\mu](x)}{\partial x_2} = \frac{1}{2\pi}\frac{\pm\mu(d)}{|x-x(d)|}\cos\psi(x,x(d)) + \Omega_2(x),$$

$$\sin\psi(x,x(d)) = \frac{x_2 - x_2(d)}{|x-x(d)|}, \quad \cos\psi(x,x(d)) = \frac{x_1 - x_1(d)}{|x-x(d)|},$$

$$|\Omega_j(x)| \le \text{const}\cdot\ln\frac{1}{|x-x(d)|}, \qquad j = 1,2;$$

the upper sign in these formulas is taken if $d = a_n^1$, while the lower sign is taken if $d = b_n^1$;
3) for $w[\mu](x)$ the following relationship holds:

$$\lim_{\epsilon\to+0}\int_{\partial S(d,\epsilon)} w[\mu](x)\frac{\partial w[\mu](x)}{\partial \mathbf{n}_x}dl = 0,$$

where the curvilinear integral of the first kind is taken over a circumference $\partial S(d,\epsilon)$, and $\mathbf{n}_x = (-\cos\psi(x,x(d)), -\sin\psi(x,x(d)))$ is a normal at the point $x \in \partial S(d,\epsilon)$, directed to the center of the circumference;

4) $|\nabla w[\mu](x)|$ belongs to $L_2(S(d,\epsilon))$ for any small $\epsilon > 0$ if and only if $\mu(d) = 0$.

Class $C^0(\overline{R^2 \setminus \Gamma^1} \setminus X)$ is defined in Remark 1 to the definition of the class $\mathbf{K}_1$ if we set $\mathcal{D} = R^2$. The proof of the theorem is based on the representation of a double-layer potential in the form of the real part of the Cauchy integral with the real density $\mu(\sigma)$:

$$w[\mu](x) = -\mathrm{Re}\,\Phi(z), \quad \Phi(z) = \frac{1}{2\pi i}\int_{\Gamma^1}\mu(\sigma)\frac{dt}{t-z}, \quad z = x_1 + ix_2,$$

where $t = t(\sigma) = (y_1(\sigma) + iy_2(\sigma)) \in \Gamma^1$. If $\mu(\sigma) \in C^{1,\lambda}(\Gamma^1)$, then for $z \notin \Gamma^1$:

$$\frac{d\Phi(z)}{dz} = -w'_{x_1} + iw'_{x_2} = -\frac{1}{2\pi i}\left(\sum_{n=1}^{N_1}\left\{\frac{\mu(b_n^1)}{t(b_n^1)-z}\frac{\mu(a_n^1)}{t(a_n^1)-z}\right\}\right.$$

$$\left. - \int_{\Gamma^1}\frac{e^{-i\alpha(\sigma)}\mu'(\sigma)}{t-z}dt\right).$$

Points I, II.1, and II.2 of Theorem 2 follow from these formulas and from the properties of Cauchy integrals, presented in [Mu68]. Points II.3 and II.4 can be proved by direct verification by using points I, II.1, and II.2.

We will construct the solution to the problem $\mathbf{D}_1$ with the assumption that $F^+(s), F^-(s) \in C^{1,\lambda}(\Gamma^1)$, $\lambda \in (0,1]$, $F(s) \in C^0(\Gamma^2)$. We will look for a solution to the problem $\mathbf{D}_1$ in the form

$$u(x) = -w[F^+ - F^-](x) + v(x), \tag{19.6}$$

where $w[F^+ - F^-](x)$ is the double-layer potential (19.5), in which $\mu(\sigma) = F^+(\sigma) - F^-(\sigma)$. The potential $w[F^+ - F^-](x)$ satisfies the Laplace equation (19.2) in $\mathcal{D} \setminus \Gamma^1$ and belongs to the class $\mathbf{K}_1$ according to Theorem 2. The limiting values of $w[F^+ - F^-](x)$ on $(\Gamma^1)^\pm$ are

$$w[F^+ - F^-](x)|_{x(s)\in(\Gamma^1)^\pm} = \mp(F^+(s) - F^-(s))/2 + w[F^+ - F^-](x(s)),$$

where $w[F^+ - F^-](x(s))$ is the direct value of the potential on Γ^1.

The function $v(x)$ in (19.6) must be a solution to the following problem.

Problem D. Find a function $v(x) \in C^0(\overline{\mathcal{D}}) \cap C^2(\mathcal{D} \setminus \Gamma^1)$, which obeys the Laplace equation (19.2) in the domain $\mathcal{D} \setminus \Gamma^1$ and satisfies the boundary conditions

$v(x)|_{x(s)\in\Gamma^1} = (F^+(s) + F^-(s))/2 + w[F^+ - F^-](x(s)) = f(s),$

$v(x)|_{x(s)\in\Gamma^2} = F(s) + w[F^+ - F^-](x(s)) = f(s).$

(If $x \in \Gamma^1$, then $w[F^+ - F^-](x)$ is the direct value of the potential on Γ^1.)

If $\mathcal{D}$ is an exterior domain, then we add the following condition at infinity:

$$|v(x)| \le const, \qquad |x| = \sqrt{x_1^2 + x_2^2} \to \infty.$$

All conditions of the problem $\mathcal{D}$ have to be satisfied in a classical sense. Obviously, $w[F^+ - F^-](x(s)) \in C^0(\Gamma^2)$. It follows from [Kr08, Theorem A.1] that $w[F^+ - F^-](x(s)) \in C^{1,\lambda/4}(\Gamma^1)$ (here by $w[F^+ - F^-](x(s))$ we mean the direct value of the potential on Γ^1). So, $f(s) \in C^{1,\lambda/4}(\Gamma^1)$ and $f(s) \in C^0(\Gamma^2)$.

We will look for the function $v(x)$ in the smoothness class $\mathbf{K}$.

We say that the function $v(x)$ belongs to the smoothness class $\mathbf{K}$ if

1. $v(x) \in C^0(\overline{\mathcal{D}}) \cap C^2(\mathcal{D} \setminus \Gamma^1)$, $\quad \nabla v \in C^0\left(\overline{\mathcal{D} \setminus \Gamma^1} \setminus \Gamma^2 \setminus X\right)$, where X is a point set consisting of the endpoints of Γ^1;
2. in a neighborhood of any point $x(d) \in X$ for some constants $\mathcal{C} > 0$, $\delta > -1$, the inequality $|\nabla v| \leq \mathcal{C}|x - x(d)|^\delta$ holds, where $x \to x(d)$ and $d = a_n^1$ or $d = b_n^1$, $n = 1, \ldots, N_1$.

The definition of the functional class $C^0\left(\overline{\mathcal{D} \setminus \Gamma^1} \setminus \Gamma^2 \setminus X\right)$ is given in Remark 1 to the definition of the smoothness class $\mathbf{K}_1$. Clearly, $\mathbf{K} \subset \mathbf{K}_1$, i.e., if $v(x) \in \mathbf{K}$, then $v(x) \in \mathbf{K}_1$.

It can be verified directly that if $v(x)$ is a solution to the problem $\mathbf{D}$ in the class $\mathbf{K}$, then the function (19.6) is a solution to the problem $\mathbf{D}_1$.

Theorem 3. *Let $\Gamma \in C^{2,\lambda/4}$, $f(s) \in C^{1,\lambda/4}(\Gamma^1)$, $\lambda \in (0,1]$, $f(s) \in C^0(\Gamma^2)$. Then the solution to the problem $\mathbf{D}$ in the smoothness class $\mathbf{K}$ exists and is unique.*

Theorem 3 has been proved in the following papers:
1) in [Kr00-1], [Kr00-2] if $\mathcal{D}$ is an interior domain;
2) in [Kr98] if $\mathcal{D}$ is an exterior domain and $\Gamma^2 \neq \emptyset$;
3) in [Kr97], [Kr05] if $\Gamma^2 = \emptyset$ and so $\mathcal{D} = R^2$ is an exterior domain.
In all these papers, the integral representations for the solution to the problem $\mathbf{D}$ in the class $\mathbf{K}$ are obtained in the form of potentials, densities in which are defined from the uniquely solvable Fredholm integro-algebraic equations of the second kind and index zero. Uniqueness of a solution to the problem $\mathbf{D}$ is proved either by the maximum principle or by the method of energy (integral) identities. In the latter case we take into account that a solution to the problem belongs to the class $\mathbf{K}$. Note that the problem $\mathbf{D}$ is a particular case of more general boundary value problems studied in [Kr00-2], [Kr98], [Kr97], [Kr05].

Note that Theorem 3 holds if $\Gamma \in C^{2,\lambda}$, $F^+(s), F^-(s) \in C^{1,\lambda}(\Gamma^1)$, $\lambda \in (0,1]$, $F(s) \in C^0(\Gamma^2)$. From Theorems 2, 3 we obtain the solvability of the problem $\mathbf{D}_1$.

Theorem 4. *Let $\Gamma \in C^{2,\lambda}$, $F^+(s), F^-(s) \in C^{1,\lambda}(\Gamma^1)$, $\lambda \in (0,1]$, $F(s) \in C^0(\Gamma^2)$. Then a solution to the problem $\mathbf{D}_1$ exists and is given by the formula (19.6), where $v(x)$ is a unique solution to the problem $\mathbf{D}$ in the class $\mathbf{K}$, ensured by Theorem 3.*

Remark 2. Let us check that the solution to the problem $\mathbf{D}_1$ given by formula (19.6) satisfies condition (19.1). Let $d = a_n^1$ or $d = b_n^1$ $(n = 1, ..., N_1)$ with r small enough; then substituting (19.6) in the integral in (19.1) we obtain

$$\int_{\partial S(d,r)} u(x)\frac{\partial u(x)}{\partial \mathbf{n}_x}dl = \int_{\partial S(d,r)} w(x)\frac{\partial w(x)}{\partial \mathbf{n}_x}dl - \int_{\partial S(d,r)} w(x)\frac{\partial v(x)}{\partial \mathbf{n}_x}dl$$
$$- \int_{\partial S(d,r)} v(x)\frac{\partial w(x)}{\partial \mathbf{n}_x}dl + \int_{\partial S(d,r)} v(x)\frac{\partial v(x)}{\partial \mathbf{n}_x}dl.$$

If $r \to 0$, then the first term tends to zero by Theorem 2 (II.3). As mentioned above, $v(x) \in \mathbf{K} \subset \mathbf{K}_1$; therefore, condition (19.1) holds for the function $v(x)$, so the fourth term tends to zero as $r \to 0$. The second term tends to zero as $r \to 0$, since $w(x)$ is bounded at the ends of Γ^1 according to Theorem 2 (I), and since $v(x)$ satisfies condition 2) in the definition of the class $\mathbf{K}$. Noting that $v(x)$ is continuous at the ends of Γ^1 owing to the definition of the class $\mathbf{K}$, and using Theorem 2 (II.2) for calculation of $\dfrac{\partial w(x)}{\partial \mathbf{n}_x}$ in the third term, we deduce that the third term tends to zero when $r \to 0$ as well. Consequently, the equality (19.1) holds for the solution to the problem $\mathbf{D}_1$ constructed in Theorem 4.

Uniqueness of a solution to the problem $\mathbf{D}_1$ follows from Theorem 1. The solution to the problem $\mathbf{D}_1$ found in Theorem 4 is, in fact, a classical solution. Let us discuss under which conditions this solution to the problem $\mathbf{D}_1$ is not a weak solution.

19.4 Nonexistence of a Weak Solution

Let $u(x)$ be a solution to the problem $\mathbf{D}_1$ defined in Theorem 4 by the formula (19.6). Consider a disc $S(d,\epsilon)$ with the center in the point $x(d) \in X$ and of radius $\epsilon > 0$ $(d = a_n^1$ or $d = b_n^1$, $n = 1,...,N_1)$. In doing so, ϵ is a fixed positive number, which can be taken small enough. Since $v(x) \in \mathbf{K}$, we have $v(x) \in L_2(S(d,\epsilon))$ and $|\nabla v(x)| \in L_2(S(d,\epsilon))$ (this follows from the definition of the smoothness class $\mathbf{K}$). Let $x \in S(d,\epsilon)$ and $x \notin \Gamma^1$. It follows from (19.6) that $|\nabla w[\mu](x)| \leq |\nabla u(x)| + |\nabla v(x)|$, whence $|\nabla w[\mu](x)|^2 \leq |\nabla u(x)|^2 + |\nabla v(x)|^2 + 2|\nabla u(x)| \cdot |\nabla v(x)| \leq 2(|\nabla u(x)|^2 + |\nabla v(x)|^2)$. Assume that $|\nabla u(x)|$ belongs to $L_2(S(d,\epsilon))$; then, integrating this inequality over $S(d,\epsilon)$, we obtain $\|\nabla w\|^2|_{L_2(S(d,\epsilon))} \leq 2(\|\nabla u\|^2|_{L_2(S(d,\epsilon))} + \|\nabla v\|^2|_{L_2(S(d,\epsilon))})$. Consequently, if $|\nabla u(x)| \in L_2(S(d,\epsilon))$, then $|\nabla w| \in L_2(S(d,\epsilon))$. However, according to Theorem 2, if $F^+(d) - F^-(d) \neq 0$, then $|\nabla w|$ does not belong to $L_2(S(d,\epsilon))$. Therefore, if $F^+(d) \neq F^-(d)$, then our assumption that $|\nabla u| \in L_2(S(d,\epsilon))$ does not hold, i.e., $|\nabla u| \notin L_2(S(d,\epsilon))$. Thus, if among numbers $a_1^1,...,a_{N_1}^1, b_1^1,...,b_{N_1}^1$ there exists such a number d that $F^+(d) \neq F^-(d)$, then for some $\epsilon > 0$ we have $|\nabla u| \notin L_2(S(d,\epsilon)) = L_2(S(d,\epsilon) \setminus \Gamma^1)$, so $u \notin W_2^1(S(d,\epsilon) \setminus \Gamma^1)$, where W_2^1 is a Sobolev space of functions from L_2, which have generalized derivatives from L_2. We have proved the following.

Theorem 5. *Let the conditions of Theorem 4 hold, and among numbers*

$$a_1^1, .., a_{N_1}^1, b_1^1, ..., b_{N_1}^1$$

there exists such a number d, that $F^+(d) \neq F^-(d)$. Then the solution to the problem $\mathbf{D}_1$, ensured by Theorem 4, does not belong to $W_2^1(S(d,\epsilon) \setminus \Gamma^1)$ for some $\epsilon > 0$, whence it follows that it does not belong to $W_{2,loc}^1(\mathcal{D} \setminus \Gamma^1)$. Here $S(d,\epsilon)$ is a disc of a radius ϵ with the center in the point $x(d) \in X$.

By $W_{2,loc}^1(\mathcal{D} \setminus \Gamma^1)$ we denote a class of functions, which belong to W_2^1 on any bounded subdomain of $\mathcal{D} \setminus \Gamma^1$. If the conditions of Theorem 5 hold, then the unique solution to the problem $\mathbf{D}_1$, constructed in Theorem 4, does not belong to $W_{2,loc}^1(\mathcal{D} \setminus \Gamma^1)$, and so it is not a weak solution. We arrive at the following corollary.

Corollary 1. *Let the conditions of Theorem 5 hold. Then a weak solution to the problem $\mathbf{D}_1$ in the class of functions $W_{2,loc}^1(\mathcal{D} \setminus \Gamma^1)$ does not exist.*

Remark 3. Even if the number d, mentioned in Theorem 5, does not exist, the solution $u(x)$ to the problem $\mathbf{D}_1$, ensured by Theorem 4, may not be a weak solution to the problem $\mathbf{D}_1$. A Hadamard example of nonexistence of a weak solution to a harmonic Dirichlet problem in a disc with continuous boundary data is given in [So88, Section 12.5] (the classical solution exists in this example).

Clearly, $L_2(\mathcal{D} \setminus \Gamma^1) = L_2(\mathcal{D})$, since Γ^1 is a set of zero measure.

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20

On Different Quasimodes for the Homogenization of Steklov-Type Eigenvalue Problems

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20.1 Introduction

Roughly speaking, a *quasimode* for an operator with a discrete spectrum on a Hilbert space can be defined as a pair $(\widetilde{w},\mu)$, where $\widetilde{w}$ is a function approaching a certain linear combination of eigenfunctions associated with the eigenvalues of the operator in a "small interval" $[\mu - r, \mu + r]$. The *remainder* r also deals with the discrepancies between $\widetilde{w}$ and the eigenfunctions.

The value of the quasimodes in describing asymptotics for low and high frequency vibrations in certain singularly perturbed spectral problems, which depend on a small parameter ε, has been made clear recently in many papers. We refer to [Pe08] for an abstract general framework that can be applied to several problems of spectral perturbation theory and to [LoPe03] and [SaSa89] for a large variety of these problems. As a matter of fact, for these problems, the spaces and the operators under consideration depend on the parameter of perturbation, and the function $\widetilde{w}$ and the numbers μ and r arising in the definition of a quasimode can also depend on this parameter.

In this chapter we deal with the low frequencies for the homogenization of a Steklov-type eigenvalue problem. Namely, we deal with harmonic functions in a bounded domain Ω of $\mathbb{R}^2$ and periodic alternating boundary conditions of Dirichlet and Steklov on a part of the boundary, namely on Σ. ε measures the periodicity of the structure. The model is of interest in geophysics, for instance: see [BuIo06], [CaDa04], [IoDa02], and [Pe07].

In what follows we construct other quasimodes $(\widetilde{w}^\varepsilon,\mu^\varepsilon)$, with $\mu^\varepsilon = O(\varepsilon^{-1})$ on different spaces from those in [Pe07]. This construction involves a new formulation of the spectral problem (20.5) in functional spaces of traces of functions on the part of the boundary where the Steklov-type conditions are imposed. The value of the new quasimodes, as initial data, in the associated second order evolution problems, is that they allow us to obtain estimates of the time t in which *standing waves* of the type $e^{\mathrm{i}\sqrt{\mu^\varepsilon}t}\widetilde{w}^\varepsilon$ approach the solutions $\mathbf{u}^\varepsilon(t)$ of the evolution problems (cf. [Pe08] and [LoPe09]). These estimates

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depend on the discrepancies between the quasimodes and the eigenelements of the spectral problem. For the sake of brevity, in this chapter, we only provide the estimates for the discrepancies between quasimodes and eigenelements and refer to [LoPe09] for estimates depending on the time parameter.

The structure of this chapter is as follows. In Section 20.1.1, we provide the general abstract framework for spaces and operators depending on the perturbation parameter ε. In Section 20.2, we introduce the Steklov eigenvalue problem under consideration (see (20.5)). We also introduce the quasimodes constructed in [Pe07] and estimates for the discrepancies of these quasimodes and the eigenelements of the spectral problem (see Theorem 1 and (20.7)). Using these results, and considering spaces of traces, in Section 20.3 we construct new quasimodes and obtain estimates for the discrepancies with the eigenelements of the corresponding spectral problem (20.2).

20.1.1 The General Abstract Framework

Let us consider ε a small parameter $\varepsilon \in (0,1)$. Let $\mathcal{V}^\varepsilon$ and $\mathcal{H}^\varepsilon$ be two separable Hilbert spaces and $\mathcal{V}^\varepsilon \subset \mathcal{H}^\varepsilon$, with dense and compact imbedding. Let $a^\varepsilon(u,v)$ be a sesquilinear, hermitian, continuous, and coercive form on $\mathcal{V}^\varepsilon$. We consider $\mathcal{V}^\varepsilon$ equipped with the scalar product inducted by $a^\varepsilon(.,.)$, namely $< u, v >_{\mathcal{V}^\varepsilon} = a^\varepsilon(u,v)$. Let $A^\varepsilon \in \mathcal{L}(\mathcal{V}^\varepsilon, (\mathcal{V}^\varepsilon)')$ be the operator associated with the form a^ε, namely, $a^\varepsilon(u,v) = < A^\varepsilon u, v >_{(\mathcal{V}^\varepsilon)' \times \mathcal{V}^\varepsilon}$.

Let us assume that

$$\|u\|_{\mathcal{H}^\varepsilon} \leq C\|u\|_{\mathcal{V}^\varepsilon}, \quad \forall u \in \mathcal{V}^\varepsilon, \tag{20.1}$$

where C is a constant independent of u and ε. Let us consider the associated spectral problem: to find λ^ε and $u^\varepsilon \in \mathcal{V}^\varepsilon$, $u^\varepsilon \neq 0$ satisfying

$$a^\varepsilon(u^\varepsilon, v) = \lambda^\varepsilon (u^\varepsilon, v)_{\mathcal{H}^\varepsilon}, \quad \forall v \in \mathcal{V}^\varepsilon. \tag{20.2}$$

Let $A^\varepsilon_{\mathcal{H}^\varepsilon}$ be the operator restriction of A^ε to $\mathcal{H}^\varepsilon$, with domain of definition $D(A^\varepsilon_{\mathcal{H}^\varepsilon}) = \{v \in \mathcal{V}^\varepsilon \,/\, A^\varepsilon v \in \mathcal{H}^\varepsilon\}$. Then, $\mathcal{A}^\varepsilon = (A^\varepsilon_{\mathcal{H}^\varepsilon})^{-1}$, $\mathcal{A}^\varepsilon : \mathcal{H}^\varepsilon \longrightarrow \mathcal{H}^\varepsilon$ is a linear, self-adjoint, positive, and compact operator on $\mathcal{H}^\varepsilon$. The eigenvalues of A^ε (respectively $\mathcal{A}^\varepsilon$) are $\{\lambda_i^\varepsilon\}_{i=1}^\infty$ (respectively $\{(\lambda_i^\varepsilon)^{-1}\}_{i=1}^\infty$), and the associated eigenfunctions are $\{u_i^\varepsilon\}_{i=1}^\infty$ which form an orthogonal basis in $\mathcal{H}^\varepsilon$ and $\mathcal{V}^\varepsilon$, u_i^ε of norm 1 in $\mathcal{H}^\varepsilon$ and of norm $\sqrt{\lambda_i^\varepsilon}$ in $\mathcal{V}^\varepsilon$.

Also, for the sake of brevity, we shall refer to pairs $(\widetilde{w}^\varepsilon, \mu^\varepsilon)$ as *quasimodes of problem (20.2)* with the remainder r^ε instead of quasimodes of the associated operators A^ε or $\mathcal{A}^\varepsilon$, which avoids specifying estimates in spaces either $\mathcal{H}^\varepsilon$ or $\mathcal{V}^\varepsilon$. Below we establish the closeness in the space $\mathcal{H}^\varepsilon \times \mathbb{R}$ ($\mathcal{V}^\varepsilon \times \mathbb{R}$, respectively) of the eigenelements of the spectral problem (20.2) to a given quasimode (cf. [OlSh92] and [Pe08] for general references and for details when applying the results to singularly perturbed spectral problems):

Given a quasimode $(\widetilde{w}^\varepsilon, \mu^\varepsilon)$ *for problem (20.2) with remainder* r^ε*,* $(\widetilde{w}^\varepsilon, \mu^\varepsilon)$ *belonging to* $\mathbf{H}^\varepsilon \times \mathbb{R}$*,* $\|\widetilde{w}^\varepsilon\|_{\mathbf{H}^\varepsilon} = 1$*, in each interval* $[\mu^\varepsilon - r^{*,\varepsilon}, \mu^\varepsilon + r^{*,\varepsilon}]$ *containing* $[\mu^\varepsilon - r^\varepsilon, \mu^\varepsilon + r^\varepsilon]$ *there are eigenvalues of (20.2),* $\{\mu^\varepsilon_{i(r^{*,\varepsilon})+k}\}_{k=1,2,\cdots,I(r^*)}$

for some index $i(r^{*,\varepsilon})$ and some natural number $I(r^{*,\varepsilon}) \geq 1$. In addition, there is $u^{*,\varepsilon} \in \mathbf{H}^\varepsilon$, $u^{*,\varepsilon}$ belonging to the eigenspace associated with all the eigenvalues in the interval $[\mu^\varepsilon - r^{*,\varepsilon}, \mu^\varepsilon + r^{*,\varepsilon}]$, satisfying

$$\|u^{*,\varepsilon}\|_{\mathbf{H}^\varepsilon} \leq C_1 \quad \textit{and} \quad \|\widetilde{w}^\varepsilon - u^{*,\varepsilon}\|_{\mathbf{H}^\varepsilon} \leq C_2 \, \frac{2r^\varepsilon}{r^{*,\varepsilon}}. \tag{20.3}$$

Here C_1 and C_2 are constants independent of ε, and the space $\mathbf{H}^\varepsilon$ can be taken to be either $\mathcal{H}^\varepsilon$ or $\mathcal{V}^\varepsilon$ depending on the operator under consideration. Also, depending on this operator, μ_j^ε can denote λ_j^ε or $(\lambda_j^\varepsilon)^{-1}$ or even rescaled eigenvalues.

20.2 The Homogenization of the Steklov Problem

Let Ω be an open bounded domain of $\mathbb{R}^{2+}$ with a Lipschitz boundary $\partial\Omega$. This boundary $\partial\Omega$ is assumed to be in contact with the line $\{x_2 = 0\}$, $\partial\Omega = \overline{\Sigma} \cup \overline{\Sigma}_f \cup \overline{\Gamma}_\Omega$, where the part of $\partial\Omega$ in contact $\{x_2 = 0\}$ is assumed to be the union of Σ_f and Σ, $\Sigma \neq \emptyset$ and $\overline{\Sigma}_f = (\overline{\Omega} \cap \{x_2 = 0\}) - \Sigma$. Without any restriction, we can assume $\Sigma = (-1/2, 1/2) \times \{0\}$ which we shall identify with the interval $(-1/2, 1/2)$ if no confusion arises. In the same way, in the case where $\Sigma_f \neq \emptyset$, we can assume that $\overline{\Sigma} \cap \overline{\Gamma}_\Omega = \emptyset$.

For fixed ε, $\varepsilon \in (0,1)$, we consider Σ to be the union of segments Σ_k^ε of length ε which we define as follows: For $k = 0, \pm1, \pm2, \pm3, \ldots, \pm N_\varepsilon$, let T_k^ε (Σ_k^ε, G_k^ε, respectively) be the homothetic T^1 (Σ^1, G^1, respectively), of ratio ε; centered at the point $\tilde{x}_k = (k\varepsilon P, 0)$. Here, T^1 and Σ^1 are segments centered at the origin, $T^1 \Subset \Sigma^1$, $G^1 = \Sigma^1 \times (0, \infty)$, ε is a small parameter that we shall make to go to zero, P is a fixed number, $P > 0$, and $2N_\varepsilon + 1$ denotes the number of Σ_k^ε contained in Σ, $N_\varepsilon = O(\varepsilon^{-1})$.

If no confusion arises, we shall write $\bigcup T^\varepsilon$ ($\bigcup \Sigma^\varepsilon$, $\bigcup G^\varepsilon$, respectively) to denote $\bigcup_{i=-N_\varepsilon}^{N_\varepsilon} T_i^\varepsilon$ ($\bigcup_{i=-N_\varepsilon}^{N_\varepsilon} \Sigma_i^\varepsilon$, $\bigcup_{i=-N_\varepsilon}^{N_\varepsilon} G_i^\varepsilon$, respectively). Also, it is self-evident that for each fixed k the change of variable

$$y = \frac{x - \tilde{x}_k}{\varepsilon} \tag{20.4}$$

transforms T_k^ε, Σ_k^ε, and G_k^ε into T^1, Σ^1, and G^1, respectively.

Let us consider the spectral problem

$$\left\{ \begin{array}{rcl} -\Delta u^\varepsilon &=& 0 \text{ in } \Omega\,, \\ u^\varepsilon &=& 0 \text{ on } \partial\Omega \setminus \bigcup T^\varepsilon, \\ \frac{\partial u^\varepsilon}{\partial x_2} + \beta^\varepsilon u^\varepsilon &=& 0 \text{ on } \bigcup T^\varepsilon\,, \end{array} \right. \tag{20.5}$$

whose variational formulation reads: Find β^ε and $u^\varepsilon \in \mathbf{V}^\varepsilon$, $u^\varepsilon \neq 0$, satisfying

$$\int_\Omega \nabla u^\varepsilon . \nabla v \, dx = \beta^\varepsilon \int_{\bigcup T^\varepsilon} u^\varepsilon v \, dx_1\,, \quad \forall v \in \mathbf{V}^\varepsilon\,. \tag{20.6}$$

Here, $\mathbf{V}^\varepsilon$ denotes the space completion of $\{v \in \mathcal{D}(\overline{\Omega}) \,/\, v = 0 \text{ on } \partial\Omega \setminus \bigcup T^\varepsilon\}$ with the norm

$$\|v\|_\varepsilon^2 = \int_\Omega |\nabla v|^2 \, dx \,. \tag{20.7}$$

The elements of $\mathbf{V}^\varepsilon$ vanish on $\Gamma_\Omega \cup \Sigma_f \cup (\Sigma \setminus \bigcup T^\varepsilon)$ (namely, on $\partial\Omega \setminus \bigcup T^\varepsilon$), and they satisfy

$$\int_\Sigma u^2 \, dx_1 = \int_{\bigcup T^\varepsilon} u^2 \, dx_1 \leq C\varepsilon \int_\Omega |\nabla u|^2 \, dx \,, \quad \forall u \in \mathbf{V}^\varepsilon, \tag{20.8}$$

where C is a constant independent of ε and u (cf. [Pe07]).

For fixed ε, the problem (20.6) can be written as an eigenvalue problem for a nonnegative, self-adjoint, compact operator $\mathbf{A}^\varepsilon$ on the space $\mathbf{V}^\varepsilon$ as follows: Find μ^ε ($\mu^\varepsilon = 1/\beta^\varepsilon$) and $u^\varepsilon \in \mathbf{V}^\varepsilon$, $u^\varepsilon \neq 0$ satisfying

$$\mathbf{A}^\varepsilon u^\varepsilon = \mu^\varepsilon u^\varepsilon, \quad \text{where} \quad < \mathbf{A}^\varepsilon u, v >= \int_{\bigcup T^\varepsilon} uv \, dx_1, \quad \forall u, v \in \mathbf{V}^\varepsilon \,. \tag{20.9}$$

Now, the eigenvalue 0 has the associated eigenspace

$$\mathrm{Ker}(\mathbf{A}^\varepsilon) = \{u \in \mathbf{V}^\varepsilon / u = 0 \text{ on } \bigcup T^\varepsilon\} \equiv H_0^1(\Omega), \tag{20.10}$$

and the rest of the spectrum, which is discrete, is denoted by $\{(\beta_i^\varepsilon)^{-1}\}_{i=1}^\infty$, where $\{\beta_i^\varepsilon\}_{i=1}^\infty$ are the set of eigenvalues with finite multiplicity of (20.6), $\beta_i^\varepsilon \to \infty$ as $i \to \infty$, with the convention of repeated indices.

Let $\{u_i^\varepsilon\}_{i=1}^\infty$ be the set of associated eigenfunctions which are assumed to be orthonormal in $\mathbf{V}^\varepsilon$. They form an orthonormal basis in the space complement orthogonal to $\mathrm{Ker}(\mathbf{A}^\varepsilon)$ in $\mathbf{V}^\varepsilon$. This orthogonal space identifies with the functions of $\mathbf{V}^\varepsilon$ which are harmonic functions in Ω, namely,

$$\mathrm{Ker}(\mathbf{A}^\varepsilon)^\perp \subset \{u \in H^1(\Omega) \,/\, \Delta u = 0 \text{ in } \Omega, \text{ and } u = 0 \text{ on } \partial\Omega \setminus \bigcup T^\varepsilon\} \,. \tag{20.11}$$

The minimax principle allows us to assert that $\beta_i^\varepsilon = O(\varepsilon^{-1})$, and results in [Pe07] show that the limit behavior of the rescaled eigenvalues $\beta_i^\varepsilon \varepsilon$ and the associated eigenfunctions is involved with the first eigenelement of the *local problem* (20.12).

The eigenvalue local problem in the half-band G^1 is: Find $(\beta^0, V^0) \in \mathbb{R}^+ \times \mathbf{V}^1$, $V^0 \neq 0$, satisfying

$$\int_{G^1} \nabla_y V^0 . \nabla_y V \, dy = \beta^0 \int_{T^1} V^0 V \, dy_1 \,, \quad \forall V \in \mathbf{V}^1 \,. \tag{20.12}$$

Here y is the *local variable* defined by (20.4), and $\mathbf{V}^1$ denotes the space completion of $\{V \in \mathcal{D}(\overline{G^1}),\, V = 0 \text{ on } \Sigma^1 \setminus T^1, V(y_1, y_2) \text{ is } y_1\text{-}\textit{periodic} \text{ in } G^1\}$ with the norm generated by the scalar product

$$(U, V)_{\mathbf{V}^1} = \int_{G^1} \nabla_y U . \nabla_y V \, dy \,.$$

As is known, the solutions V^0 of (20.12) are harmonic functions in G^1 satisfying $V^0(y) \to c_{V^0}$ as $y_2 \to +\infty$, where c_{V_0} is an unknown but well-determined constant and (20.12) has a discrete spectrum. We refer to [Pe07] for details of proofs.

20.2.1 The Construction Quasimodes for (20.5)

For each eigenfunction V^0 of (20.12), $\|V^0\|_{\mathbf{V}^1} = 1$, let $w^\varepsilon(x)$ be the function defined by

$$w^\varepsilon(x_1, x_2) = V^0(y_1, y_2) \quad \text{for } (x_1, x_2) \in G_0^\varepsilon = \varepsilon G^1 \tag{20.13}$$

and extended by periodicity to all the half-bands G_i^ε such that the corresponding Σ_i^ε are contained in Σ. For simplicity, without any restriction, we can assume that the T_i^ε do not cut the extremes $x_1 = \pm 1/2$ of the interval $\overline{\Sigma} = [-1/2, 1/2]$ (cf. [Pe07] in this connection).

Let us consider the cutoff function η^ε,

$$\eta^\varepsilon(x) = \eta\left(x_2 {\delta_\varepsilon}^{-1}\right), \tag{20.14}$$

where $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$ and η is a smooth function with a compact support, $supp\,(\eta') \subset [\frac{1}{3}, \frac{2}{3}]$,

$$\eta \in C^1(\mathbb{R}), \quad 0 \le \eta \le 1\,, \quad \eta(t) = 1 \text{ for } t \le \frac{1}{3} \quad \text{and} \quad \eta(t) = 0 \text{ for } t \ge \frac{2}{3}\,.$$

For each fixed $V^0(y)$ solution of (20.12), the function δ_ε can be chosen to be

$$\delta_\varepsilon = \tilde{k}\varepsilon|\ln\varepsilon|\,, \tag{20.15}$$

where $\tilde{k}$ is a constant which depends on V^0. More specifically, for any fixed positive integer J, $J \ge 2$, we can determine $\tilde{k}$ depending on V^0 and J, namely $\tilde{k} = \tilde{k}(V^0, J)$, that ensures the existence of a constant $\tilde{C} = \tilde{C}(V^0)$ and of $\varepsilon_J > 0$, ε_J depending on V^0 and J, such that the estimates

$$|V^0(y) - c_{V^0}| \le \tilde{C}\varepsilon^J \quad \text{and} \quad |\nabla_y V^0(y)| \le \tilde{C}\varepsilon^J \tag{20.16}$$

hold for $\varepsilon < \varepsilon_J$ and $y_2 > \delta_\varepsilon 3^{-1}\varepsilon^{-1}$. We refer to [Pe07] and [PaPe07] for proofs.

Let us denote by $w^\varepsilon\eta^\varepsilon = w^\varepsilon(x)\eta^\varepsilon(x)$ the function $V^0(x/\varepsilon)$ extended by periodicity to all the Σ_i^ε contained in Σ and multiplied by the function $\eta^\varepsilon(x)$ which is only dependent on x_2. $w^\varepsilon\eta^\varepsilon$ is a periodic function of the x_1 variable which vanishes on $\Sigma \setminus \bigcup T^\varepsilon$. It also vanishes for $x_2 > (2/3)\tilde{k}\varepsilon\,|\ln\varepsilon|$ and takes the value of $w^\varepsilon(x)$ for $0 \le x_2 \le (1/3)\tilde{k}\varepsilon\,|\ln\varepsilon|$.

Let Ω_Σ be the domain $\Omega \cap (\Sigma \times (0, \infty))$. In the case where $\Sigma_f = \emptyset$ it is clear that $\Omega_\Sigma = \Omega$. But, even if we assume that $\Omega_\Sigma = \Omega$, we cannot assert that $w^\varepsilon\eta^\varepsilon \in \mathbf{V}^\varepsilon$ since it does not necessarily vanish near $\overline{\Gamma}_\Omega \cap \{x_2 = 0\}$. From this function we construct another which satisfies this condition also in the case where $\Sigma_f \neq \emptyset$.

For any fixed intervals (a,b) and (c,d) contained in Σ, $(a,b) \Subset (c,d)$ (i.e., $(a,b) \Subset (c,d) \subset (-1/2,1/2)$), let ψ be a function

$$\psi \in C_0^\infty(\mathbb{R}), \quad 0 \le \psi \le 1, \quad \psi(x_1) = 1 \text{ if } x \in [a,b], \quad \psi(x_1) = 0 \text{ if } x_1 \notin (c,d).$$

Then, we define the *boundary layer function* $w^\varepsilon \eta^\varepsilon \psi$, concentrating its support in a small region near Σ,

$$(w^\varepsilon \eta^\varepsilon \psi)(x) = w^\varepsilon(x_1,x_2)\eta^\varepsilon(x_2)\psi(x_1). \tag{20.17}$$

Obviously, $w^\varepsilon \eta^\varepsilon \psi \in \mathbf{V}^\varepsilon$, where now the function $\eta^\varepsilon \psi \in C_0^\infty(\mathbb{R}^2)$ satisfies:

$$\eta^\varepsilon \psi(x) = \begin{cases} 1 & \text{if} \quad (x_1,x_2) \in [a,b] \times [0,(1/3)\tilde{k}\varepsilon\,|\ln \varepsilon|] \\ \psi(x_1) & \text{if} \quad 0 \le x_2 \le (1/3)\tilde{k}\varepsilon\,|\ln \varepsilon| \\ \eta^\varepsilon(x_2) & \text{if} \quad a \le x_1 \le b \\ 0 & \text{if} \quad x_2 \ge (2/3)\tilde{k}\varepsilon\,|\ln \varepsilon| \\ 0 & \text{if} \quad (x_1,x_2) \in \overline{\Omega}, \text{ and } x_1 \notin \Sigma. \end{cases}$$

Note that from the definition of $w^\varepsilon \eta^\varepsilon \psi$, we can assume that w^ε is extended by periodicity over the whole half-plane $\mathbb{R}^{2+}$. We gather bounds and properties for w^ε in Proposition 1 in Section 20.3. Some of these properties, namely, estimates (20.22), (20.23), and (20.24), are used in [Pe07] to prove the results in the following theorem.

Theorem 1. *Let (β^0, V^0) be any eigenelement of (20.12), V^0 with norm 1 in $\mathbf{V}^1$ (that is, $\int_{G^1} |\nabla_y V^0|^2\,dy = 1$). There exists a sequence d^ε, $d^\varepsilon \to 0$, as $\varepsilon \to 0$, such that there are eigenvalues β^ε of (20.6) with $\varepsilon\beta^\varepsilon \in [\beta^0 - d^\varepsilon, \beta^0 + d^\varepsilon]$ (or equivalently, such that $(\beta^\varepsilon)^{-1} \in [\varepsilon(\beta^0)^{-1} - r^\varepsilon, \varepsilon(\beta^0)^{-1} + r^\varepsilon]$ for $r^\varepsilon = O(d^\varepsilon\varepsilon)$).*

In addition, there are $\tilde{u}^\varepsilon$, with $\int_\Omega |\nabla \tilde{u}^\varepsilon|^2\,dx = 1$, $\tilde{u}^\varepsilon$ in the eigenspace of all the eigenfunctions u^ε of (20.6) associated with the eigenvalues β^ε such that $\varepsilon\beta^\varepsilon \in [\beta^0 - \tilde{d}^\varepsilon, \beta^0 + \tilde{d}^\varepsilon]$ (or equivalently, such that $(\beta^\varepsilon)^{-1} \in [\varepsilon(\beta^0)^{-1} - \tilde{r}^\varepsilon, \varepsilon(\beta^0)^{-1} + \tilde{r}^\varepsilon]$ for $\tilde{r}^\varepsilon = O(\tilde{d}^\varepsilon\varepsilon)$, $\varepsilon(\beta^0)^{-1} > \tilde{r}^\varepsilon$), with $\tilde{d}^\varepsilon \to 0$ and $d^\varepsilon/\tilde{d}^\varepsilon \to 0$ as $\varepsilon \to 0$ (or equivalently, $\tilde{r}^\varepsilon \to 0$ and $r^\varepsilon/\tilde{r}^\varepsilon \to 0$ as $\varepsilon \to 0$), and $\tilde{u}^\varepsilon$ satisfying

$$\int_\Omega |\nabla(\tilde{u}^\varepsilon - \alpha^\varepsilon w^\varepsilon \eta^\varepsilon \psi)|^2\,dx \le C\Big(\frac{r^\varepsilon}{\tilde{r}^\varepsilon}\Big)^2, \tag{20.18}$$

where α^ε is the constant

$$\alpha^\varepsilon = \left(\int_\Omega |\nabla(w^\varepsilon \eta^\varepsilon \psi)|^2\,dx\right)^{-1/2}, \tag{20.19}$$

C is a constant independent of ε, and the functions $w^\varepsilon \eta^\varepsilon \psi$ are defined in (20.13)–(20.17). The sequences d^ε and r^ε can be taken as follows:

$$d^\varepsilon = K_1|\ln \varepsilon|^{-1/2}, \quad \text{and} \quad r^\varepsilon = K_2\varepsilon|\ln \varepsilon|^{-1/2}, \tag{20.20}$$

where K_1, K_2 are certain constants independent of ε. Also, sequences $\tilde{d}^\varepsilon$ and $r^\varepsilon/\tilde{r}^\varepsilon = d^\varepsilon/\tilde{d}^\varepsilon$ can be chosen in order to get either smaller intervals $[\beta^0 - \tilde{d}^\varepsilon, \beta^0 + \tilde{d}^\varepsilon]$ or improved bounds (20.18).

Moreover, considering $\varepsilon(\beta^0)^{-1} > \tilde{r}^\varepsilon$ and (20.20), possible choices of $\tilde{r}^\varepsilon$ are

$$\tilde{r}^\varepsilon = K_3\varepsilon|\ln \varepsilon|^{-\beta}, \quad \text{with } K_3 \text{ any constant and } 0 < \beta < \frac{1}{2}\,. \tag{20.21}$$

In particular, $d^\varepsilon/\tilde{d}^\varepsilon = r^\varepsilon/\tilde{r}^\varepsilon = O(|\ln \varepsilon|^{-1/4})$ is one of these possible choices.

Remark 1. We refer to [Pe07] for the proof of Theorem 1 and further results obtained from the statement. As a matter of fact, applying results in Theorem 1 allows us to assert that each eigenvalue β^0 of (20.12) is an accumulation point of the rescaled eigenvalues $\varepsilon\beta^\varepsilon$ of (20.6). In addition, as regards the eigenfunctions, for any eigenfunction V^0 associated with the eigenvalue β^0 of (20.12), and sufficiently small ε, functions $\alpha^\varepsilon(w^\varepsilon\eta^\varepsilon\psi)$ are called the *quasimodes* of (20.5), approaching linear combinations of eigenfunctions $\widetilde{u}^\varepsilon$ in small intervals, as stated in Theorem 1. The norm used for the approach is (20.7), and considering the support of $(w^\varepsilon\eta^\varepsilon\psi)$, we can assert that these eigenfunctions $\{\tilde{u}^\varepsilon\}_{\varepsilon>0}$ concentrate their support asymptotically in a thin layer of width $O(\varepsilon|\ln \varepsilon|)$ around a part of the boundary Σ (in which the *supp*(ψ) is contained) and they vanish outside.

20.3 The Modified Quasimodes for (20.5)

The aim of this section is to construct quasimodes for problem (20.6) from those in Section 20.2, which involve spaces, forms, operators, and evolution problems derived from the framework in Section 20.1.1.

It should be emphasized that formulation (20.6) [(20.5), respectively] in $\mathbf{V}^\varepsilon$ does not amount to (20.2). For the sake of brevity here we only outline forms and spaces arising in the framework (20.1) and (20.2) for (20.5). We refer to [Gr92] for details of definitions of spaces of traces and to [LoPe09] for proofs.

We first assume that $\Sigma_f \neq \emptyset$ and we denote $\overline{\Gamma}_0 = \partial\Omega \cap \{x_2 = 0\}$. Then, we consider $\mathcal{V}^\varepsilon$ the space formed by the traces on Γ_0 of the elements of $\mathrm{Ker}(\mathbf{A}^\varepsilon)^\perp$ (see definition (20.11)), which is an eigenspace of $\widetilde{H}^{1/2}(\Gamma_0)$ whose elements vanish outside $\bigcup T^\varepsilon$. Let us define $\mathcal{H}^\varepsilon$ the space completion of $\mathcal{V}^\varepsilon$ in $L^2(\Gamma_0)$. Then, we define

$$a^\varepsilon(f,g) = \langle A^\varepsilon f, g\rangle_{(\mathcal{V}^\varepsilon)'\times\mathcal{V}^\varepsilon}\ , \quad \forall f, g \in \mathcal{V}^\varepsilon$$

where A^ε is the operator from $\mathcal{V}^\varepsilon$ into $(\mathcal{V}^\varepsilon)'$ defined by $A^\varepsilon f = \chi_{\bigcup T^\varepsilon} \frac{\partial U^f}{\partial x_2}\big|_{\Gamma_0}$, U^f being the element of $\mathrm{Ker}(\mathbf{A}^\varepsilon)^\perp$, such that $U^f|_{\Gamma_0} = f$, and $\chi_{\bigcup T^\varepsilon}$ the characteristic function of the set $\bigcup T^\varepsilon$. With the notation above, it can be verified that the eigenelements $(u^\varepsilon, \beta^\varepsilon)$ of (20.6) provide the eigenelements $(u^\varepsilon|_{\Gamma_0}, \beta^\varepsilon)$ of (20.2).

In order to obtain estimates for the discrepancies between the quasimodes and the eigenfunctions in Theorem 2 below, we introduce some properties for the functions w^ε defined by (20.13) in the following proposition (cf. [Pe07] for the proof of some of these properties and further references on the technique).

Proposition 1. *Let w^ε be the functions defined by (20.13) and extended by periodicity to $\mathbb{R}^{2+}$, $w^\varepsilon \in H^1_{loc}(\mathbb{R}^{2+})$. They satisfy the estimates*

$$\varepsilon \|\nabla w^\varepsilon\|^2_{L^2(\Omega_\Sigma)} \leq C(V^0)\,, \tag{20.22}$$

$$\|w^\varepsilon\|^2_{L^2(\Omega_\Sigma)} \leq C(V^0)\,, \tag{20.23}$$

$$\|w^\varepsilon\|^2_{L^2(\Omega_\Sigma \cap \{x_2<\varepsilon\})} \leq \varepsilon C(V^0)\,, \tag{20.24}$$

for sufficiently small ε, where $C(V^0)$ is a constant independent of ε.

In addition, for the functions $w^\varepsilon \eta^\varepsilon \psi$ in (20.17) and the constants α^ε in (20.19), and for sufficiently small ε, we have

$$\tilde{C}_1(V^0)\varepsilon^{-1} \leq \|\nabla(w^\varepsilon \eta^\varepsilon \psi)\|^2_{L^2(\Omega)} \leq \tilde{C}_2(V^0)\varepsilon^{-1}\,, \tag{20.25}$$

and

$$\tilde{c}_1(V^0)\sqrt{\varepsilon} \leq \alpha^\varepsilon \leq \tilde{c}_2(V^0)\sqrt{\varepsilon}\,, \tag{20.26}$$

where $\tilde{C}_1(V^0)$, $\tilde{C}_2(V^0)$, $\tilde{c}_1(V^0)$, and $\tilde{c}_2(V^0)$ are strictly positive constants independent of ε.

Proof. Estimates (20.22)–(20.24) and the left-hand side of (20.25) have been proved in [Pe07], while (20.26) is a direct consequence of (20.19) and (20.25). Therefore, the right-hand side of (20.25) is yet to be proved.

Since $\operatorname{supp}(\psi) \subset \Sigma$, the integrals below affect only Ω_Σ. We also assume that C and C_{V^0} always denote certain constants independent of ε.

$$\begin{aligned}\|\nabla(w^\varepsilon \eta^\varepsilon \psi)\|^2_{L^2(\Omega)} &= \int_{\Omega_\Sigma} |\nabla(w^\varepsilon \eta^\varepsilon \psi)|^2\, dx \\ &\leq C\left(\int_{\Omega_\Sigma} |\nabla w^\varepsilon|^2 (\eta^\varepsilon \psi)^2\, dx + \int_{\Omega_\Sigma} (w^\varepsilon)^2\, |\nabla(\eta^\varepsilon \psi)|^2\, dx\right).\end{aligned}$$

From (20.22) and because $(\eta^\varepsilon \psi)^2 \leq 1$, the first integral above is bounded by $C\, C_{V^0} \varepsilon^{-1}$. Let us consider the second integral. From the definition for η^ε and ψ, we can write

$$\begin{aligned}&\int_{\Omega_\Sigma} (w^\varepsilon)^2\, |\nabla(\eta^\varepsilon \psi)|^2\, dx \\ &\quad \leq C\left(\int_{\Omega_\Sigma} (w^\varepsilon)^2 \left|\frac{\partial \eta^\varepsilon}{\partial x_2}\right|^2 (\psi)^2 dx + \int_{\Omega_\Sigma} (w^\varepsilon)^2\, (\eta^\varepsilon)^2 \left|\frac{\partial \psi}{\partial x_1}\right|^2 dx\right).\end{aligned} \tag{20.27}$$

Now from the bounds for $|\psi'(x_1)|$ and $\eta^\varepsilon(x)$ by constants independent of ε and from (20.23) we have that the last integral on the right-hand side of (20.27) is bounded by a constant independent of ε. As for the first integral inside the brackets, we have

$$\int_{\Omega_\Sigma} (w^\varepsilon)^2 \left|\frac{\partial \eta^\varepsilon}{\partial x_2}\right|^2 (\psi)^2 dx \leq C\frac{1}{(\delta_\varepsilon)^2} \int_{\Omega_\Sigma \cap \{x_2 \in [\delta_\varepsilon/3, 2\delta_\varepsilon/3]\}} (w^\varepsilon(x))^2\, dx$$

as a consequence of the bound for ψ. Now, performing the change of variable in the last integral from x to y, and taking into account the definition by periodicity of w^ε ((cf. (20.13), (20.14), (20.15), and (20.16)) we are led to write

$$\begin{aligned}\int_{\Omega_\Sigma} (w^\varepsilon)^2 \left|\frac{\partial \eta^\varepsilon}{\partial x_2}\right|^2 (\psi)^2 dx &\leq \frac{\varepsilon^2}{\delta_\varepsilon^2} \sum_{i=-N_\varepsilon}^{N_\varepsilon} \int_{G^1 \cap \{y_2 \in [\delta_\varepsilon 3^{-1}\varepsilon^{-1},\, 2\delta_\varepsilon 3^{-1}\varepsilon^{-1}]\}} |V^0(y)|^2\, dy \\ &\leq C(2N_\varepsilon+1)\frac{\varepsilon^2}{\delta_\varepsilon^2} \int_{G^1 \cap \{y_2 \in [\delta_\varepsilon 3^{-1}\varepsilon^{-1},\, 2\delta_\varepsilon 3^{-1}\varepsilon^{-1}]\}} |V^0(y) - c_{V^0}|^2\, dy \\ &\quad + C(2N_\varepsilon+1)\frac{\varepsilon^2}{\delta_\varepsilon^2} \int_{G^1 \cap \{y_2 \in [\delta_\varepsilon 3^{-1}\varepsilon^{-1},\, 2\delta_\varepsilon 3^{-1}\varepsilon^{-1}]\}} (c_{V^0})^2\, dy\,.\end{aligned}$$

Next, we prove that each integral in the above inequality is bounded by $C_{V^0}\varepsilon^{-1}$.

Indeed, on account of $N_\varepsilon = O(\varepsilon^{-1})$ (20.15), we have for the last integral

$$\begin{aligned}C(2N_\varepsilon+1)\frac{\varepsilon^2}{\delta_\varepsilon^2} \int_{G^1 \cap \{y_2 \in [\delta_\varepsilon 3^{-1}\varepsilon^{-1},\, 2\delta_\varepsilon 3^{-1}\varepsilon^{-1}]\}} (c_{V^0})^2\, dy &\leq C\frac{1}{\delta_\varepsilon} \\ &\leq C\frac{1}{\tilde{k}\varepsilon|\ln\varepsilon|} < \frac{C_{V^0}}{\varepsilon}\,.\end{aligned}$$

On the other hand, with the same argument it suffices to take $J = 2$ in (20.15) and (20.16), and sufficiently small ε, namely $\varepsilon < \varepsilon_J$, to obtain

$$C(2N_\varepsilon+1)\frac{\varepsilon^2}{\delta_\varepsilon^2} \int_{G^1 \cap \{y_2 \in [\delta_\varepsilon 3^{-1}\varepsilon^{-1},\, 2\delta_\varepsilon 3^{-1}\varepsilon^{-1}]\}} |V^0(y) - c_{V^0}|^2\, dy \leq C\frac{\varepsilon^4}{\delta_\varepsilon} < \frac{C_{V^0}}{\varepsilon}.$$

Therefore, the first integral on the right-hand side of (20.27) is bounded by $C_{V^0}\varepsilon^{-1}$ and the right-hand side of (20.25) also holds. Hence, all the inequalities in the statement of the proposition hold.

Theorem 2. *Let* (β^0, V^0) *be any eigenelement of (20.12),* V^0 *with norm* 1 *in* $\mathbf{V}^1$ *(that is,* $\int_{G^1} |\nabla_y V^0|^2\, dy = 1$*). With the same notation as in Theorem 1, let* $\widetilde{\mathrm{w}}^\varepsilon$ *be the projection of* $w^\varepsilon\eta^\varepsilon\psi$ *into* $\mathrm{Ker}(\mathbf{A}^\varepsilon)^\perp$*. Then, there are* $u^{*,\varepsilon} \in \mathrm{Ker}(\mathbf{A}^\varepsilon)^\perp$*, each* $u^{*,\varepsilon}$ *being a linear combination of the eigenfunctions associated with the eigenvalues* β^ε *such that* $\varepsilon\beta^\varepsilon \in [\beta^0 - \tilde{d}^\varepsilon, \beta^0 + \tilde{d}^\varepsilon]$

(or equivalently, such that $(\beta^\varepsilon)^{-1} \in [\varepsilon(\beta^0)^{-1} - \tilde r^\varepsilon, \varepsilon(\beta^0)^{-1} + \tilde r^\varepsilon]$), such that $\|u^{*,\varepsilon}\|_{L^2(\Sigma)} \leq \widetilde c$ and the inequality

$$\left\| u^{*,\varepsilon} - \frac{\widetilde{\mathrm w}^\varepsilon}{\|\widetilde{\mathrm w}^\varepsilon\|_{L^2(\Gamma_0)}} \right\|_{L^2(\Gamma_0)} = \left\| u^{*,\varepsilon} - \frac{\widetilde{\mathrm w}^\varepsilon}{\|\widetilde{\mathrm w}^\varepsilon\|_{L^2(\Sigma)}} \right\|_{L^2(\Sigma)} \leq \widetilde C \frac{r^\varepsilon}{\tilde r^\varepsilon} \tag{20.28}$$

hold. Here $\widetilde c$ and $\widetilde C$ are constants independent of ε, and r^ε and $\tilde r^\varepsilon$ the order functions defined by (20.20) and (20.21).

Proof. On account of (20.10), (20.11), and $\mathbf V^\varepsilon = H_0^1(\Omega) \bigoplus \mathrm{Ker}(\mathbf A^\varepsilon)^\perp$, we can consider $w^\varepsilon \eta^\varepsilon \psi = \widetilde{\mathrm w}^\varepsilon + W^\varepsilon$, where $W^\varepsilon \in H_0^1(\Omega)$ and $\widetilde{\mathrm w}^\varepsilon$ denotes the projection of $w^\varepsilon \eta^\varepsilon \psi$ into $\mathrm{Ker}(\mathbf A^\varepsilon)^\perp$. That is, $\widetilde{\mathrm w}^\varepsilon \in \mathbf V^\varepsilon$ and it is harmonic in Ω. Since the functions $\widetilde u^\varepsilon$ constructed in Theorem 1 are linear combinations of eigenfunctions of (20.6), they are already harmonic functions belonging to $\mathrm{Ker}(\mathbf A^\varepsilon)^\perp$ and the estimates (20.18) also hold, for $\widetilde u^\varepsilon$ and $\alpha^\varepsilon \widetilde{\mathrm w}^\varepsilon$. Specifying, we can write

$$\left\| (\alpha^\varepsilon)^{-1} \widetilde u^\varepsilon - \widetilde{\mathrm w}^\varepsilon \right\|_{\mathbf V^\varepsilon} \leq C \frac{r^\varepsilon}{\tilde r^\varepsilon} \frac{1}{\alpha^\varepsilon}, \tag{20.29}$$

for a certain constant C independent of ε, and for α^ε, r^ε, and $\tilde r^\varepsilon$ defined by (20.19), (20.20), and (20.21), respectively.

Let us note that, since all the functions above vanish on Σ_f, without any restriction we can consider indifferently norms in $L^2(\Gamma_0)$ or $L^2(\Sigma)$.

Then, considering (20.8) and (20.26), from (20.29) we can write

$$\left\| \widetilde u^\varepsilon (\alpha^\varepsilon)^{-1} - \widetilde{\mathrm w}^\varepsilon \right\|_{L^2(\Sigma)} \leq C \frac{r^\varepsilon}{\tilde r^\varepsilon} . \tag{20.30}$$

In order to get estimates of the type (20.3) for $u^{*,\varepsilon} = \widetilde u^\varepsilon (\alpha^\varepsilon \|\widetilde{\mathrm w}^\varepsilon\|_{L^2(\Sigma)})^{-1}$, $\widetilde w^\varepsilon = \widetilde{\mathrm w}^\varepsilon \|\widetilde{\mathrm w}^\varepsilon\|^{-1}_{L^2(\Sigma)}$, and $\mathbf H^\varepsilon = \{u \in L^2(\Gamma_0), u = 0 \text{ in } \Gamma_0 \setminus \bigcup T^\varepsilon\}$, let us consider $\|\widetilde{\mathrm w}^\varepsilon\|_{L^2(\Sigma)}$ and prove that it is bounded by some constant independent of ε. Indeed, by construction we can write

$$\|\widetilde{\mathrm w}^\varepsilon\|^2_{L^2(\Sigma)} = \|\widetilde{\mathrm w}^\varepsilon + W^\varepsilon\|^2_{L^2(\Sigma)} = \|w^\varepsilon \eta^\varepsilon \psi\|^2_{L^2(\Sigma)} = \|w^\varepsilon \psi\|^2_{L^2(\Sigma)} .$$

Consequently, for the interval (a, b) arising in the definition (20.17), we have

$$\|w^\varepsilon\|^2_{L^2(a,b)} \leq \|\widetilde{\mathrm w}^\varepsilon\|^2_{L^2(\Sigma)} \leq \|w^\varepsilon\|^2_{L^2(\Sigma)} , \tag{20.31}$$

and considering the definition of w^ε by periodicity from $V^0(y)$, the integrals on the left- and right-hand sides of the above inequalities can be replaced by

$$N_\varepsilon^1 \, \varepsilon \int_{T^1} (V^0(y_1, 0))^2 \, dy_1 \quad \text{and} \quad (2N_\varepsilon + 1) \, \varepsilon \int_{T^1} (V^0(y_1, 0))^2 \, dy_1 ,$$

respectively, where N_ε^1 and $(2N_\varepsilon + 1)$ are the number of T^ε contained in the segments $[a, b]$ and Σ, respectively. Since both of them are of the order $O(\varepsilon^{-1})$, for sufficiently small ε, the norm $\|\widetilde{\mathrm{w}}^\varepsilon\|_{L^2(\Sigma)}$ is bounded from below and from above by constants independent of ε. Thus, from (20.31) and (20.30), we have

$$\left\| \frac{\widetilde{u}^\varepsilon}{\alpha^\varepsilon \, \|\widetilde{\mathrm{w}}^\varepsilon\|_{L^2(\Sigma)}} \right\|_{L^2(\Sigma)} \leq \widetilde{c} \text{ and } \left\| \frac{\widetilde{u}^\varepsilon}{\alpha^\varepsilon \, \|\widetilde{\mathrm{w}}^\varepsilon\|_{L^2(\Sigma)}} - \frac{\widetilde{\mathrm{w}}^\varepsilon}{\|\widetilde{\mathrm{w}}^\varepsilon\|_{L^2(\Sigma)}} \right\|_{L^2(\Sigma)} \leq \widetilde{C} \, \frac{r^\varepsilon}{\widetilde{r}^\varepsilon} \, ,$$

for certain constants $\widetilde{C}$ and $\widetilde{c}$ independent of ε. Thus, we deduce that $u^{*,\varepsilon} = \widetilde{u}^\varepsilon(\alpha^\varepsilon \, \|\widetilde{\mathrm{w}}^\varepsilon\|_{L^2(\Sigma)})^{-1}$ satisfies (20.28), and also the rest of the properties in the statement of the theorem are satisfied. Therefore, the theorem is proved.

Remark 2. Theorem 2 provides us with the estimates (20.3) arising in the definition of quasimodes for the functions $u^{*,\varepsilon} = \widetilde{u}^\varepsilon(\alpha^\varepsilon \|\widetilde{\mathrm{w}}^\varepsilon\|_{L^2(\Sigma)})^{-1}$ and $\widetilde{w}^\varepsilon = \widetilde{\mathrm{w}}^\varepsilon \|\widetilde{\mathrm{w}}^\varepsilon\|^{-1}_{L^2(\Sigma)}$, the space $\mathbf{H}^\varepsilon = \{u \in L^2(\Gamma_0), u = 0 \text{ in } \Gamma_0 \setminus \bigcup T^\varepsilon\}$ (equivalently, $\mathbf{H}^\varepsilon = \mathcal{H}^\varepsilon$) and the numbers $\mu^\varepsilon = \beta^0 \varepsilon^{-1}$ and $\widetilde{r}^\varepsilon = r^{*,\varepsilon}$. In this connection, note that the square of the norms in $L^2(\Gamma_0)$ or $L^2(\Sigma)$ in Theorem 2 can be replaced by the square of the norm in $\prod_{i=-N_\varepsilon}^{N_\varepsilon} L^2(T_i^\varepsilon)$.

Remark 3. Also, note that without considering the normalization of the new quasimodes in $\mathbf{H}^\varepsilon$, bounds for the discrepancies $\|\widetilde{u}^\varepsilon - \alpha^\varepsilon \widetilde{\mathrm{w}}^\varepsilon\|_{\mathbf{H}^\varepsilon}$ which improve (20.28) can be obtained. Now, $\mathbf{H}^\varepsilon$ could be either the space $\mathcal{H}^\varepsilon$ or $\mathcal{V}^\varepsilon$ (cf. [LoPe09]).

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21

Asymptotic Analysis of Spectral Problems in Thick Multi-Level Junctions

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21.1 Introduction and Statement of the Problem

Spectral boundary-value problems are considered in a new kind of perturbed domain, namely, *thick multi-level junctions.* Boundary-value problems in thick one-level junctions (thick junctions) have been intensively investigated recently (see, for instance, [BlGaGr07], [BlGaMe08], [Me08] and, the references there). In [MeNa97]–[Me(3)01], classification of thick one-level junctions was given and basic results were obtained both for boundary-value and spectral problems in thick junctions of different types. It was shown that qualitative properties of solutions essentially depend on the junction type and on the conditions given on the boundaries of the attached thin domains. It is known that the asymptotic behavior of the spectrum of a perturbed spectral problem is highly sensitive to perturbation, and it is unexpected. This was also observed for spectral problems in thick junctions with Neumann conditions ([MeNa97] and [Me00]), with Dirichlet conditions ([Me99] and [Me(3)01]), with Fourier conditions ([Me(2)01]) and with Steklov ones ([Me(1)01]).

The approach of paper [Me06], where a spectral problem in a plane thick two-level junction was considered, and the abstract scheme developed in [Me(4)01] are used to investigate the spectral problem discussed here.

21.1.1 Statement of the Problem

Let B be a finite union of smooth plane domains which do not intersect or touch. In addition, the set B is strictly included in the square $\square := \{\xi' = (\xi_1, \xi_2) : 0 < \xi_1 < 1, \quad 0 < \xi_2 < 1\}$. Let us divide B into two classes: $B^{(1)} = \bigcup_{k=1}^{K_1} B_k^{(1)}$ and $B^{(2)} = \bigcup_{k=1}^{K_2} B_k^{(2)}$.

A model thick two-level junction Ω_ε consists of the junction body $\Omega_0 = \{x \in \mathbb{R}^3 : \ x' = (x_1, x_2) \in Q, \quad 0 < x_3 < \gamma(x')\}$, and a large number of thin cylinders $G_\varepsilon^{(m)} = \cup_{k=1}^{K_m} G_\varepsilon^{(m)}(k)$,

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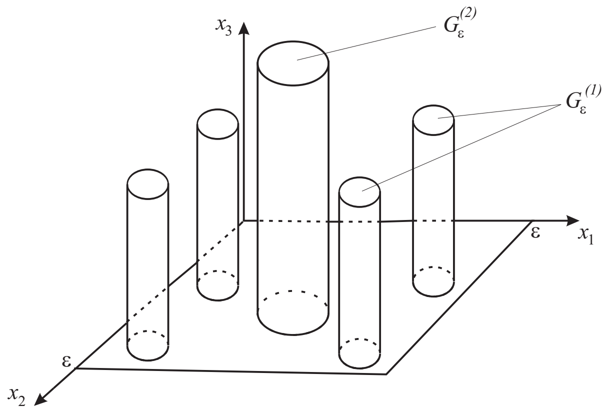

Fig. 21.1. The cell of the alternation.

$$G_\varepsilon^{(m)}(k) = \bigcup_{i,j=0}^{N-1} \big\{ x : \ (\tfrac{x_1}{\varepsilon} - i, \tfrac{x_2}{\varepsilon} - j) \in B_k^{(m)}, \ x_3 \in (-d_m, 0] \big\}, \quad m = 1, 2.$$

In this case $Q = (0,a) \times (0,a)$, γ is a smooth and a-periodic function, $\min_{x' \in \overline{Q}} \gamma(x') = \gamma_0 > 0$, N is a large natural number, and $\varepsilon = a/N$ is a small discrete parameter that characterizes the distance between nearby thin cylinders and their thickness; $0 < d_2 \le d_1$. Thus, $\Omega_\varepsilon = \Omega_0 \bigcup G_\varepsilon, \ G_\varepsilon = G_\varepsilon^{(1)} \bigcup G_\varepsilon^{(2)}$.

The thin cylinders G_ε are divided into two levels $G_\varepsilon^{(1)}$ and $G_\varepsilon^{(2)}$ depending on their length, and they are ε-periodically alternated along the Ox_1-direction and Ox_2-direction and they are joined with Ω_0 over the ε-homothetic images $\varepsilon\big((i,j) + B_k^{(1)}\big)$, $i, j = 0, 1, \ldots, N-1$, $k = 1, \ldots, K_1$, and $\varepsilon\big((i,j) + B_k^{(2)}\big)$, $i, j = 0, 1, \ldots, N-1$, $k = 1, \ldots, K_2$, of the classes $B^{(1)}$ and $B^{(2)}$, respectively. A cell of the alternation is shown in Figure 21.1.

In Ω_ε we consider the following spectral problem:

$$\begin{array}{rcll}
-\Delta_x \, u^\varepsilon(x) & = & \lambda(\varepsilon)\, u^\varepsilon(x), & x \in \Omega_\varepsilon; \\
\partial_\nu u^\varepsilon(x) & = & -\varepsilon\, \kappa_1\, u^\varepsilon(x), & x \in S_\varepsilon^{(1)}; \\
\partial_\nu u^\varepsilon(x) & = & -\varepsilon\, \kappa_2\, u^\varepsilon(x), & x \in S_\varepsilon^{(2)}; \\
\partial_{x_1}^p u^\varepsilon|_{x_1=0} & = & \partial_{x_1}^p u^\varepsilon|_{x_1=a}, & (x_2, x_3) \in (0,a) \times (0, \gamma(0, x_2)), \ p = 0, 1; \\
\partial_{x_2}^p u^\varepsilon|_{x_2=0} & = & \partial_{x_2}^p u^\varepsilon|_{x_2=a}, & (x_1, x_3) \in (0,a) \times (0, \gamma(x_1, 0)), \ p = 0, 1; \\
\partial_\nu u^\varepsilon(x) & = & 0, & x \in \Gamma_\varepsilon,
\end{array} \tag{21.1}$$

with the Fourier conditions (κ_1, κ_2 are positive constants) on $S_\varepsilon^{(m)}$ (the union of the lateral surfaces of the cylinders $G_\varepsilon^{(m)}$ from the mth level, $m = 1, 2$), with the periodic condition on the lateral faces Γ_0 of the junction body Ω_0, and with the Neumann conditions on $\Gamma_\varepsilon = \partial\Omega_\varepsilon \setminus \big(S_\varepsilon^{(1)} \cup S_\varepsilon^{(2)} \cup \Gamma_0\big)$.

The aim is to study the asymptotic behavior of the spectrum of problem (21.1) and corresponding eigenfunctions as $\varepsilon \to 0$, i.e., when the number of attached thin cylinders from each level infinitely increases and their thickness vanishes.

21.2 Special Integral Identities and Extension Operators

To homogenize boundary-value problems in thick multi-structures with nonhomogeneous Neumann, Fourier, or nonlinear conditions on the boundaries of the thin attached domains, the method of special integral identities was proposed in [Me(1)01], [Me(2)01], [Me08]. Following [Me(2)01], [Me08], for the 1-periodic continuations with respect to ξ_1 and ξ_2 of the solutions $Y_k^{(m)}$, $k = 1, \dots, K_m,\ m = 1, 2$, of the problems

$$\Delta_\xi Y_k^{(m)}(\xi) = \frac{p_k^{(m)}}{|B_k^{(m)}|},\ \ \xi \in B_k^{(m)};\quad \partial_\nu Y_k^{(m)} = 1,\ \ \xi \in \partial B_k^{(l)};\quad \langle Y_k^{(m)} \rangle_{B_k^{(m)}} = 0,$$

where $\langle Y_k^{(m)} \rangle_{B_k^{(m)}} = \int_{B_k^{(m)}} Y_k^{(m)}(\xi) d\xi$, we derive the integral identities

$$\varepsilon \int_{S_\varepsilon^{(m)}} v d\sigma_x = \sum_{k=1}^{K_m} \int_{G_\varepsilon^{(m)}(k)} \Big(\frac{p_k^{(m)}}{|B_k^{(m)}|}\, v \,+\, \varepsilon \nabla_\xi Y_k^{(m)}|_{\xi = \frac{x'}{\varepsilon}} \cdot \nabla_{x'} v \Big) dx \qquad (21.2)$$

for all $v \in H^1\big(G_\varepsilon^{(m)}\big), m = 1, 2$. Here $|B_k^{(m)}|$, $p_k^{(m)}$ are the area and perimeter of the two-dimensional domain $B_k^{(m)}$.

In $\mathcal{H}_\varepsilon := \{u \in H^1(\Omega_\varepsilon) :\ \ u$ is a-periodic on $\Gamma_0\}$ we define the norm $\|\cdot\|_{\varepsilon,k_1,k_2}$ that is generated by the following scalar product:

$$\langle u, v \rangle_{\varepsilon,\kappa_1,\kappa_2} = \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v \, dx \,+\, \varepsilon\, \kappa_1 \int_{S_\varepsilon^{(1)}} u\, v \, d\sigma_x \,+\, \varepsilon\, \kappa_2 \int_{S_\varepsilon^{(2)}} u\, v\, d\sigma_x.$$

It is easy to prove that the norms $\|\cdot\|_{H^1(\Omega_\varepsilon)}$ and $\|\cdot\|_{\varepsilon,\kappa_1,\kappa_2}$ are uniformly equivalent with respect to ε. Define $A_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{H}_\varepsilon$ by the following equality:

$$\langle A_\varepsilon u, v \rangle_{\varepsilon,\kappa_1,\kappa_2} = \int_{\Omega_\varepsilon} u(x)\, v(x)\, dx \quad \forall\ u, v \in \mathcal{H}_\varepsilon. \qquad (21.3)$$

Obviously, operator A_ε is self-adjoint, positive, and compact. Thus, problem (21.1) is equivalent to the spectral problem $A_\varepsilon u = \lambda^{-1}(\varepsilon)\, u$ in $\mathcal{H}_\varepsilon$ and for each fixed $\varepsilon > 0$ there is a sequence of eigenvalues

$$0 < c_0 \le \lambda_1(\varepsilon) \le \cdots \le \lambda_n(\varepsilon) \le \cdots \to +\infty, \qquad (21.4)$$

and a sequence of the eigenfunctions $\{u_n^\varepsilon\}$: $\big(u_n^\varepsilon, u_m^\varepsilon\big)_{L_2(\Omega_\varepsilon)} = \delta_{n,m}$, $n, m \in \mathbb{N}$.

By the minimax principle for eigenvalues, we have $\lambda_n(\varepsilon) \le C_1(n)$; then with the help of (21.2) we get $\|u_n^\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C_2(n)$ for any $n \in \mathbb{N}$.

Using the scheme of construction of extension operators (see [Me(4)01]) and integral identities (21.2), we can prove the following theorem.

Theorem 1. *There exist extension operators* $\mathbf{P}_\varepsilon^{(m,k)} : H^1\big(\Omega_0 \cup G_\varepsilon^{(m)}(k)\big) \mapsto H^1(\Omega_m)$ *such that* $\forall\ n \in \mathbb{N}\ \ \exists C > 0\ \ \exists \varepsilon_0 > 0\ \ \forall\ \varepsilon \in (0, \varepsilon_0)$*:*

$$\sum_{m=1}^{2} \sum_{k=1}^{K_m} \| \, \mathbf{P}_\varepsilon^{(m,k)} u_n^\varepsilon \, \|_{H^1(\Omega_m)} \le C \ \| \, u_n^\varepsilon \, \|_{H^1(\Omega_\varepsilon)},$$

where $\overline{\Omega_m} = \overline{\Omega_0} \cup \overline{D_m}$, $\ D_m = Q \times (-d_m, 0)$, $\ m = 1, 2$.

21.3 Convergence Theorem and Homogenized Problem

With the help of the extension operators constructed in Theorem 1 and identities (21.2) we establish the following convergences.

Theorem 2. *Let* $\lambda(\varepsilon)$ *be an eigenvalue of problem (21.1) and let* u^ε *be the corresponding eigenfunction whose* $\|u^\varepsilon\|_{L^2(\Omega_\varepsilon)} = 1$. *Let* $\lambda(\varepsilon) \to \mu_0$, $\ u^\varepsilon|_{\Omega_0} \to v_0^+$ *weakly in* $H^1(\Omega_0)$, *and for each* $m = 1, 2$ *and* $k = 1, \ldots, K_m$ *the restriction* $\big(\mathbf{P}_\varepsilon^{(m,k)} u^\varepsilon\big)|_{D_m} \to v_0^{m,k}$ *weakly in* $H^1(D_m)$ *as* $\varepsilon \to 0$.

Then μ_0 *is an eigenvalue and the multi-sheeted function* $\mathbf{v}_0$ *such that* $\mathbf{v}_0|_{\Omega_0} = v_0^+$, $\mathbf{v}_0|_{D_m} = v_0^{m,k}$, $m = 1, 2$, $k = 1, \ldots, K_m$, *is the corresponding eigenfunction of the homogenized spectral problem*

$$\begin{cases} -\Delta_x \, v_0^+ = \mu_0 \, v_0^+ \quad \text{in } \Omega_0, \\ v_0^+ \text{ is } a\text{-periodic on } \Gamma_0, \quad \partial_\nu v_0^+ = 0 \quad \text{on } \partial\Omega_0 \setminus (\Gamma_0 \cup Q), \\ -|B_k^{(m)}| \, \partial^2_{x_3} v_0^{m,k} + \kappa_m \, p_k^{(m)} \, v_0^{m,k} = \mu_0 \, |B_k^{(m)}| \, v_0^{m,k} \quad \text{in } D_m, \\ v_0^{m,k}|_{x_3=0} = v_0^+|_{x_3=0}, \qquad \big(\partial_{x_3} v_0^{m,k}\big)|_{x_3=-d_m} = 0, \\ m = 1, 2, \ k = 1, \ldots, K_m, \\ \sum_{m=1}^{2} \sum_{k=1}^{K_m} |B_k^{(m)}| \, \partial_{x_3} v_0^{m,k}(x', 0) = \partial_{x_3} v_0^+(x', 0) \quad \text{on } Q. \end{cases} \tag{21.5}$$

We write $\mathcal{V}_0 := L^2(\Omega_0) \times \underbrace{L^2(D_1) \times \cdots \times L^2(D_1)}_{K_1} \times \underbrace{L^2(D_2) \times \cdots \times L^2(D_2)}_{K_2}$

with the inner product

$$\big(\mathbf{u}, \mathbf{v}\big)_{\mathcal{V}_0} = \int_{\Omega_0} u_0 v_0 \, dx + \sum_{m=1}^{2} \sum_{k=1}^{K_m} |B_k^{(m)}| \int_{D_m} u_k^{(m)} v_k^{(m)} \, dx,$$

where

$$\mathbf{u} = \big(u_0, u_1^{(1)}, \ldots, u_{K_1}^{(1)}, u_1^{(2)}, \ldots, u_{K_2}^{(2)}\big)$$

and

$$\mathbf{v} = \big(v_0, v_1^{(1)}, \ldots, v_{K_1}^{(1)}, v_1^{(2)}, \ldots, v_{K_2}^{(2)}\big).$$

Define the Hilbert space $\mathcal{H}_0 = \{\mathbf{u} \in \mathcal{V}_0 : \ u_0 \in H^1(\Omega_0), \ u_0$ is a-periodic on Γ_0; $\exists\ \partial_{x_3} u_k^{(m)} \in L^2(D_m)$ and $u_0|_Q = u_k^{(m)}|_Q$ for any $m = 1, 2, \ k = 1, \ldots, K_m\}$ with the scalar product

$$\begin{aligned}\big(\mathbf{u}, \mathbf{v}\big)_{\mathcal{H}_0} &= \int_{\Omega_0} \nabla u_0 \cdot \nabla v_0 \, dx \\ &+ \sum_{m=1}^{2} \sum_{k=1}^{K_m} \int_{D_m} \Big(|B_k^{(m)}| \, \partial_{x_3} u_k^{(m)} \, \partial_{x_3} v_k^{(m)} \ + \ \kappa_m \, p_k^{(m)} \, u_k^{(m)} \, v_k^{(m)} \Big) dx.\end{aligned}$$

Problem (21.5) is equivalent to the spectral problem $A_0 \mathbf{v}_0 = \mu_0^{-1} \mathbf{v}_0$ in $\mathcal{H}_0$, where the operator $A_0 : \mathcal{H}_0 \mapsto \mathcal{H}_0$ is defined by the equality

$$\big(A_0 \mathbf{u}, \mathbf{v}\big)_{\mathcal{H}_0} = \big(\mathbf{u}, \mathbf{v}\big)_{\mathcal{V}_0} \quad \forall\ \mathbf{u}, \mathbf{v} \in \mathcal{H}_0; \tag{21.6}$$

obviously, it is self-adjoint, positive, and continuous, but noncompact. From Theorem 2 and (21.4) it follows that the spectrum of A_0 is situated in $[c_0, +\infty)$.

We assume that $\Theta_k^{(m)} \le c_0$ for each $m = 1, 2$ and $k = 1, \ldots, K_m$, where $\Theta_k^{(m)} = \frac{k_m \, p_k^{(m)}}{|B_k^{(m)}|}$. The other cases can be considered similarly as in [Me06].

Solving the differential equations of problem (21.5) in D_m and taking the first transmission condition $v_0^{m,k}|_{x_3=0} = v_0^+|_{x_3=0}$ and the boundary condition $\big(\partial_{x_3} v_0^{m,k}\big)|_{x_3=-d_m} = 0$ into account, we obtain

$$v_0^{m,k}(x) = \frac{v_0^+(x', 0)}{\cos\big(d_m \sqrt{\mu_0 - \Theta_k^{(m)}}\big)} \cos\left(\sqrt{\mu_0 - \Theta_k^{(m)}}(x_3 + d_m)\right).$$

Substituting these representations into the second transmission condition, we get the nonlinear spectral problem $L(\mu) v_0^+ = 0$ in $H_\sharp^1(\Omega_0) = \{v \in H^1(\Omega_0) :$ v is a-periodic on $\Gamma_0\}$ $\ (\mu \in [c_0, +\infty))$ for the operator function

$$L(\mu) := (\mu + 1)\, A_1 + \sum_{m=1}^{2} \sum_{k=1}^{K_m} |B_k^{(m)}| \sqrt{\mu - \Theta_k^{(m)}} \ \tan\Big(d_m \sqrt{\mu - \Theta_k^{(m)}}\Big) A_2 - \mathbb{I},$$

where $A_1, \ A_2$ are self-adjoint, compact operators in $H_\sharp^1(\Omega_0)$ such that, for $\forall\ \varphi, \psi \in H_\sharp^1(\Omega_0)$,

$$(A_1 \varphi, \psi)_{H_\sharp^1(\Omega_0)} = \int_{\Omega_0} \varphi(x) \psi(x) dx, \quad (A_2 \varphi, \psi)_{H_\sharp^1(\Omega_0)} = \int_Q \varphi(x', 0) \psi(x', 0) dx'.$$

Theorems on existence and concentration of the spectrum for such self-adjoint operator-functions and minimax principles for the eigenvalues were proved in [Me94], [HrMe96]. From these results we have the following theorem.

Theorem 3. *The spectrum of L consists of normal eigenvalues and the left accumulation points $\{P_t : \ t \in \mathbb{N}\}$ that are poles of the functions*

$$\tan\Big(d_m\sqrt{\mu - \Theta_k^{(m)}}\Big), \quad m = 1, 2, \quad k = 1, \ldots, K_m. \tag{21.7}$$

These points divide the eigenvalues into the sequences

$$c_0 \le \mu_1^{(1)} \le \cdots \le \mu_n^{(1)} \le \cdots \to \ P_1,$$
$$P_{t-1} < \mu_1^{(t)} \le \cdots \le \mu_n^{(t)} \le \cdots \to \ P_t \quad as \quad n \to \infty.$$

21.4 Asymptotic Approximations

Let $\mathbf{v}_0 \in \mathcal{H}_0$ and μ be a solution to problem (21.5). Using the method of matched asymptotic expansions, we construct an approximation $R_\varepsilon \in \mathcal{H}_\varepsilon$:

$$R_\varepsilon(x) = v_0^+(x) + \varepsilon\chi_0(x_3)\sum_{i=1}^{3}\left(Z_i(\xi) - \delta_{i,3}\xi_3\right)|_{\xi=\frac{x}{\varepsilon}}\,\partial_{x_i}v_0^+(x', 0), \quad x \in \Omega_0,$$

$$R_\varepsilon = v_0^{m,k} + \varepsilon\sum_{i=1}^{2} Y_i(\xi_i)|_{\xi_i=\frac{x_i}{\varepsilon}}\partial_{x_i}v_0^{m,k}$$
$$+\,\varepsilon\,\chi_0\sum_{i=1}^{3}\big(Z_i(\xi) - Y_i(\xi_i)\big)|_{\xi=\frac{x}{\varepsilon}}\partial_{x_i}v^+(x', 0)$$

on $G_\varepsilon^{(m)}(k)$, $m = 1, 2, \ k = 1, \ldots, K_m$. Here χ_0 is a smooth cut-off function that equals 1 in a neighborhood of zero; $\{Z_i\}$ are 1-periodic in ξ_1 and ξ_2 $(\xi_3 > 0)$ junction-layer solutions to the following problems:

$$\left\{\begin{array}{rcll} -\Delta_\xi Z_i(\xi) & = & 0, & \xi \in \Pi, \\ \partial_{\xi_3} Z_i(\xi', 0) & = & 0, & (\xi', 0) \in \partial\Pi^+ \setminus B, \\ \partial_{\nu_{\xi'}} Z_i & = & -\delta_{1,i}\nu_1(\xi') - \delta_{2,i}\nu_2(\xi'), & \xi \in \partial\Pi^- \setminus B, \end{array}\right. \tag{21.8}$$

where $\delta_{i,k}$ is the Kronecker symbol, $\Pi = \Pi^+ \cup \Pi^-$, $\Pi^+ = \square \times (0, +\infty)$, $\Pi^- = \big(\cup_{k=1}^{K_1}\Pi_k^{1,-}\big) \cup \big(\cup_{k=1}^{K_2}\Pi_k^{2,-}\big)$, $\Pi_k^{m,-} = B_k^{(m)} \times (-\infty, 0]$. We reassign the semi-infinite cylinders $\{\Pi_k^{m,-}\}_{m=1,2,\ k=1,\ldots,K_m}$ and sets $\{B_k^{(m)}\}_{m=1,2,\ k=1,\ldots,K_m}$ by $\{\Pi_j^-\}_{j=1,\ldots,K}$ and $\{B_j\}_{j=1,\ldots,K}$, respectively, $K = K_1 + K_2$.

Lemma 1. *There exist K solutions to the junction-layer problem (21.8) at $i = 3$, which have the following differentiable asymptotics:*

$$\Xi_j(\xi) = \left\{\begin{array}{ll} \xi_3 + \mathcal{O}(\exp(-\gamma_3^+\,\xi_3)), & \xi_3 \to +\infty, \ \xi \in \Pi^+, \\ \frac{\xi_3}{|B_j|} + \alpha_j + \mathcal{O}(\exp(\gamma_j^-\,\xi_3)), & \xi_3 \to -\infty, \ \xi \in \Pi_j^-, \\ \alpha_j^{(k)} + \mathcal{O}(\exp(\gamma_k^-\,\xi_3)), & \xi_3 \to -\infty, \ \xi \in \Pi_k^-, \quad k \ne j, \end{array}\right.$$

where $\gamma_j^{\pm}$ *are some positive constants. Any other solution to problem (21.8)* $(i=3)$, *which has polynomial growth at infinity, can be presented as a linear combination* $\beta_0 + \sum_{j=1}^{K} \beta_j \varXi_j$.

There exists a unique solution Z_i *to problem (21.8)* $(i = 1, 2)$ *with the following asymptotics:*

$$Z_i(\xi) = \begin{cases} \mathcal{O}(\exp(-\gamma_i^{+}\,\xi_3)), & \xi_3 \to +\infty,\ \xi \in \varPi^{+}, \\ -\xi_i + b_j^{(i)} + \mathcal{O}(\exp(-\gamma_{i,j}^{-}\,\xi_3)), & \xi_3 \to -\infty,\ \xi \in \varPi_j^{-}. \end{cases}$$

We take Z_3 as a linear combination $(1-\beta_2-\cdots-\beta_K)\varXi_1(\xi)+\beta_2\varXi_2(\xi)+\cdots+\beta_K\varXi_K(\xi)$; $\beta_2,\ldots,\beta_K$ are found from the matching conditions. The functions Y_1, Y_2, Y_3 are 1-periodic with respect to ξ_1, ξ_2; $Y_i(\xi_i) = -\xi_i + b_j^{(i)}$, $\xi \in \varPi_j^{-}$, $j = 1,\ldots,K$, $i = 1, 2$; Y_3 is equal to the polynomial part of $(1-\beta_2-\cdots-\beta_K)\varXi_1(\xi)+\beta_2\varXi_2(\xi)+\cdots+\beta_K\varXi_K(\xi)$ on the cell of periodicity $\varPi^{-}$.

Substituting R_ε and μ_0 into problem (21.1) and finding residuals, we get

$$\|\, R_\varepsilon \,-\, \mu_0 A_\varepsilon R_\varepsilon \,\|_{\mathcal{H}_\varepsilon} \,\le\, c(\delta)\,\varepsilon^{1-\delta} \quad (\delta > 0). \tag{21.9}$$

21.4.1 Approximation near the Essential Spectrum.

Let $\mu_0 \in \sigma_{ess}(A_0) = \{P_t : \ t \in \mathbb{N}\}$, i.e., μ_0 coincides with one of the poles of the functions (21.7) at $m_0 \in \{1, 2\}$ and $k_0 \in \{1,\ldots,K_{m_0}\}$. Fix one cylinder $G_{i_0 j_0}^{(m_0,k_0)}(\varepsilon) = \big\{x : \ (\frac{x_1}{\varepsilon} - i_0, \frac{x_2}{\varepsilon} - j_0) \in B_{k_0}^{(m_0)}, \ x_3 \in (-d_{m_0}, 0]\big\}$ from the set $G_\varepsilon^{(m_0)}(k_0)$ and construct the following approximation:

$$W_\varepsilon(x) = \begin{cases} \alpha(\varepsilon)\, \cos\Big(\sqrt{\mu_0 - \varTheta_{k_0}^{(m_0)}}\,(x_3 + d_{m_0})\Big), & x \in G_{i_0 j_0}^{(m_0,k_0)}(\varepsilon), \\ 0, \quad x \in \varOmega_\varepsilon \setminus G_{i_0 j_0}^{(m_0,k_0)}(\varepsilon). \end{cases} \tag{21.10}$$

Here we choose $\alpha(\varepsilon)$ such that $\|W_\varepsilon\|_{\mathcal{H}_\varepsilon} = 1$. Substituting $\{W_\varepsilon(\cdot),\ \mu_0\}$ into problem (21.1) in place of $\{u(\varepsilon,\cdot),\ \lambda(\varepsilon)\}$ and finding residuals, we get

$$\|W_\varepsilon \,-\, \mu_0\, A_\varepsilon\big(W_\varepsilon\big)\|_{\mathcal{H}_\varepsilon} \,\le\, c\,\varepsilon^{\frac{1}{4}}. \tag{21.11}$$

21.5 Justification of the Asymptotics

To justify the asymptotic approximations constructed above, we use the scheme proposed in [Me(4)01] for investigation of the asymptotic behavior $(\epsilon \to 0)$ of eigenvalues and eigenvectors of a family of operators $\{A_\epsilon : H_\epsilon \mapsto H_\epsilon\}_{\epsilon>0}$ losing compactness in the limit passage. This scheme generalizes the procedure of the justification of the asymptotic behavior of eigenvalues and eigenvectors of boundary-value problems in perturbed domains.

In our case this is the family of operators $\{A_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{H}_\varepsilon\}_{\varepsilon>0}$ defined in (21.3). Recall that A_ε corresponds to problem (21.1) and $A_0 : \mathcal{H}_0 \mapsto \mathcal{H}_0$, which is defined by (21.6), corresponds to the homogenized problem (21.5).

Define special coupling operators P_ε and S_ε. For better understanding, we write the diagram

$$\begin{array}{ccc} \mathcal{H}_\varepsilon & \subset\subset & \mathcal{V}_\varepsilon \\ {\scriptstyle P_\varepsilon}\big\downarrow & & \big\uparrow{\scriptstyle S_\varepsilon} \\ \mathcal{Z}_0 \subset \mathcal{H}_0 & \subset & \mathcal{V}_0 \end{array}$$

in which the imbedding $\mathcal{H} \subset \mathcal{V}$ means that the space $\mathcal{H}$ is densely and only continuously embedded into $\mathcal{V}$, but the imbedding $\mathcal{H} \subset\subset \mathcal{V}$ is also compact. Here $\mathcal{Z}_0 = \{\mathbf{u} \in \mathcal{H}_0 : \ u_k^{(m)} \in H^1(D_m), \ m = 1, 2, \ k = 1, \ldots, K_m\}$. Obviously, $\mathcal{Z}_0 \subset\subset \mathcal{V}_0$.

The operator $S_\varepsilon : \mathcal{V}_0 \mapsto \mathcal{V}_\varepsilon$ assigns to any multi-sheeted function $\mathbf{v} \in \mathcal{V}_0$ the function $S_\varepsilon \mathbf{v}$ that is equal to v_0 in Ω_0 and to $v_k^{(m)}|_{G_\varepsilon^{(m)}(k)}, \ m = 1, 2, \ k = 1, \ldots, K_m$. Clearly, S_ε is uniformly bounded with respect to ε. Thus, the condition (C1) in the scheme [Me(4)01] is satisfied.

The operator P_ε from condition (C2) is associated with the extension operators from Theorem 1, and in our case it puts every function u from $\mathcal{H}_\varepsilon$ into the respective multi-sheeted function from $\mathcal{Z}_0$.

Conditions (C3) and (C4) are verified in the proof of Theorem 2. Conditions (C5) and (C6), in fact, have been verified in the previous section. The result of the action of the operator R_ε from condition (C5) is the construction of the approximation function R_ε which satisfies the estimate (21.9). The estimate (21.11) coincides with a similar estimate from condition (C6).

Thus, all conditions (C1)–(C6) of the scheme from [Me(4)01] are satisfied for problems (21.1) and (21.5). Applying this scheme, we get the following theorems.

Theorem 4 (Hausdorff convergence). *Only points of the spectrum of problem (21.5) are accumulation points for the spectrum of problem (21.1) as* $\varepsilon \to 0$.

The eigenvalues $\{\lambda_n(\varepsilon)\}$ at fixed indices n are usually called *low eigenvalues* (see [Me(3)01]); the corresponding eigenfunctions are called *low frequency oscillations.*

Definition 1 ([Me(3)01]). *The value* $\mathcal{T} := \sup_{n\in\mathbf{N}} \limsup_{\varepsilon\to 0} \lambda_n(\varepsilon)$ *is called the threshold of the low eigenvalues of problem (21.1).*

Theorem 5 (low frequency convergence). *Let* $\{\lambda_n(\varepsilon) : n \in \mathbb{N}_0\}$ *be the ordered sequence (21.4) of eigenvalues of problem (21.1), let* $\{u_n(\varepsilon, \cdot) : n \in \mathbb{N}\}$ *be the corresponding sequence of eigenfunctions orthonormalized in* $L^2(\Omega_\varepsilon)$*, and let* $c_0 < \mu_1^{(1)} \le \cdots \le \mu_n^{(1)} \le \cdots \to P_1$ *be the first series of eigenvalues of the homogenized problem (21.5) (see Theorem 3).*

Then the threshold of the low eigenvalues of problem (21.1) is equal to P_1*, and for any* $n \in \mathbb{N}$ $\lambda_n(\varepsilon) \to \mu_n^{(1)}$ *as* $\varepsilon \to 0$*. There exists a subsequence of the sequence* $\{\varepsilon\}$ *(again denoted by* $\{\varepsilon\}$*) such that* $\mathbf{P}_\varepsilon u_n(\varepsilon, \cdot) \to \mathbf{v}_n^{(0)}$ *weakly* *in* $\mathcal{Z}_0$ *as* $\varepsilon \to 0$, *where* $\{\mathbf{v}_n^{(0)}\}$ *are the corresponding eigenfunctions of the homogenized problem (21.5) that satisfy the condition* $\big(\mathbf{v}_n^{(0)}, \mathbf{v}_m^{(0)}\big)_{\mathcal{V}_0} = \delta_{n,m}$.

Theorem 6 (asymptotic estimates for the low eigenvalues). *Let* $\mu_n^{(1)} = \mu_{n+1}^{(1)} = \cdots = \mu_{n+r-1}^{(1)}$ *be an* r*-multiple eigenvalue of problem (21.5) from the first series and let* $\mathbf{v}_n^{(1)}, \ldots, \mathbf{v}_{n+r-1}^{(1)}$ *be the corresponding eigenfunctions orthonormalized in* $\mathcal{V}_0$.

Then for any $\delta > 0$ *and* $n \in \mathbb{N}$ *and sufficiently small* ε, *we have*

$$|\lambda_n(\varepsilon) - \mu_n^{(1)}| \ \le \ c_0(n, \delta)\ \varepsilon^{1-\delta}.$$

In addition, for any $\delta > 0$ *and* $i \in \{0, 1, \ldots, r-1\}$, *there exist* $\varepsilon_0 > 0,\ C_i > 0$, *and* $\{\alpha_{ik}(\varepsilon),\ k = 0, 1, \ldots, r-1\} \subset \mathbb{R}$ *such that* $0 < c_1 < \sum_{k=0}^{r-1}(\alpha_{ik}(\varepsilon))^2 < c_2$ *and for any* $\varepsilon \in (0, \varepsilon_0)$

$$\Big\| R_\varepsilon^{(n+i)} - \sum\nolimits_{k=0}^{r-1} \alpha_{ik}(\varepsilon) u_{n+k}(\varepsilon, \cdot) \Big\|_{H^1(\Omega_\varepsilon)} \le C_i(n, \delta)\ \varepsilon^{1-\delta},$$

where $\{R_\varepsilon^{(n+i)}\}$ *is the approximation function constructed over the function* $\mathbf{v}_{n+i}^{(1)}$ *(see Section 21.4).*

It follows from Theorems 4 and 5 that there exist other converging sequences of eigenvalues $\lambda_{n(\varepsilon)}(\varepsilon) \quad (n(\varepsilon) \to +\infty \quad \text{as} \quad \varepsilon \to 0)$ that are called *high frequency convergences*; the corresponding eigenfunctions are called *high frequency oscillations.*

Theorem 7 (high frequency convergence and estimates). *Let* $\mu_n^{(t)} = \mu_{n+1}^{(t)} = \cdots = \mu_{n+r-1}^{(t)}$ *be an* r*-multiple eigenvalue of problem (21.5) from the* t*th series; the functions* $\mathbf{v}_n^{(t)}, \ldots, \mathbf{v}_{n+r-1}^{(t)}$ *are the corresponding eigenfunctions orthonormalized in* $\mathcal{V}_0$.

Then, for any $\delta > 0$, *there exist* $\varepsilon_{n,t} > 0$ *and* $c > 0$ *such that for all values of the parameter* $\varepsilon \in (0, \varepsilon_{n,t})$, *the interval*

$$I_{n,t}(\varepsilon) = \Big(\mu_n^{(t)} - c\varepsilon^{1-\delta}\ ,\ \mu_n^{(t)} + c\varepsilon^{1-\delta}\Big)$$

contains exactly r *eigenvalues of problem (21.5).*

For the approximation function $R_\varepsilon^{(n+i,t)}$ $(i = 0, 1, \ldots, r-1)$ *constructed over* $\mathbf{v}_{n+i}^{(t)}$, *the following asymptotic estimate:*

$$\Bigg\| \frac{R_\varepsilon^{(n+i,t)}}{\|R_\varepsilon^{(n+i,t)}\|_{\mathcal{H}_\varepsilon}} - \widetilde{U}_i(\varepsilon, \cdot) \Bigg\|_{\mathcal{H}_\varepsilon} \le c(n, t, \delta)\ \varepsilon^{1-\delta}, \qquad \|\widetilde{U}_i(\varepsilon, \cdot)\|_{\mathcal{H}_\varepsilon} = 1,$$

holds, where $\widetilde{U}_i(\varepsilon, \cdot)$ *is a linear combination of eigenfunctions of problem (21.1) that correspond to the eigenvalues from the interval* $I_{n,t}(\varepsilon)$.

Theorem 8 (asymptotic behavior near the essential spectrum). *Let μ_0 coincide with one of the points of the essential spectrum $\{P_t: \ t \in \mathbb{N}\}$ of the homogenized problem (21.5).*

Then there exist $c_0 > 0$ and $\varepsilon_0 > 0$ such that for all values of the parameter $\varepsilon \in (0, \varepsilon_0)$ the interval

$$\left(\frac{1}{\mu_0} \ - \ c_0\, \varepsilon^{\frac{1}{4}} \ , \ \frac{1}{\mu_0} \ + \ c_0\, \varepsilon^{\frac{1}{4}}\right)$$

contains finitely many eigenvalues of the operator A_ε.

There exists a finite linear combination $\widetilde{U}_\varepsilon$ $(\|\widetilde{U}_\varepsilon\|_\varepsilon = 1)$ of the eigenfunctions $u^\varepsilon_{k(\varepsilon)+i}$, $i = \overline{0,\ p(\varepsilon)}$, that correspond, respectively, to the eigenvalues $\big(\lambda_{k(\varepsilon)+i}(\varepsilon)\big)^{-1}$ of operator A_ε from the segment $\left[\frac{1}{\mu_0} \ - \ c_0\, \varepsilon^{\frac{1}{8}} \ , \ \frac{1}{\mu_0} \ + \ c_0\, \varepsilon^{\frac{1}{8}}\right]$, such that

$$\left\| W_\varepsilon - \widetilde{U}_\varepsilon \right\|_{\mathcal{H}_\varepsilon} \ \leq \ 2\varepsilon^{\frac{1}{8}},$$

where W_ε is defined by (21.10).

From the estimates in Theorems 6 and 7 it follows that the low and high frequency vibrations are vibrations of the junction Ω_ε like an entire system. Vibrations like W_ε (see (21.10)) are vibrations of Ω_ε, in which each cylinder can have its own frequency. Therefore, such vibrations are called *pseudovibrations* (for more details see [Me(3)01]). They appear near the essential spectrum of the homogenized problem, and their energy is concentrated on the thin cylinders.

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22

Integral Approach to Sensitive Singular Perturbations

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22.1 Introduction

The main purpose of this chapter is to give general ideas on a kind of singular perturbation arising in thin shell theory when the middle surface is elliptic and the shell is fixed on a part of the boundary and free on the rest, as well as an integral heuristic procedure reducing these problems to simpler ones. The system depends essentially on the parameter ε equal to the relative thickness of the shell. It appears that the "limit problem" for $\varepsilon = 0$ is highly ill posed. Indeed, the boundary conditions on the free boundary are not "adapted" to the system of equations; they do not satisfy the Shapiro–Lopatinskii (SL) condition. Roughly speaking, this amounts to some kind of "transparency" of the boundary conditions, which allows some kind of locally indeterminate oscillations along the boundary, exponentially decreasing inside the domain. This pathological behavior only occurs for $\varepsilon = 0$. In fact, for $\varepsilon > 0$ the problem is "classical." When ε is positive but small, the "determinacy" of the oscillations only holds with the help of boundary conditions on other boundaries, as well as the small terms coming from $\varepsilon > 0$.

In these kinds of situations, the limit problem has no solution within the classical theory of partial differential equations, which uses distribution theory. It is sometimes possible to prove the convergence of the solutions u^ε towards some limit u^0, but this "limit solution" and the topology of the convergence are concerned with abstract spaces not included in the distribution space.

After recalling the SL condition (Section 22.2), we give in Section 22.3 a very simple example of such a perturbation problem. The geometry of the domain (an infinite strip) allows explicit treatment by Fourier transform in the longitudinal direction. The inverse Fourier transform within distribution theory is only possible for $\varepsilon > 0$, whereas for $\varepsilon = 0$ it is only possible in the framework of analytic functionals (highly singular and not enjoyi localization properties). This example shows the prominent role of components with high frequency; for small ε, the "smooth parts" (i.e., with small $|\xi|$) of the solutions may be neglected with respect to "singular ones" (i.e., with large $|\xi|$). We also

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recall an example of the elliptic Cauchy problem (in fact, Hadamard's counter-example) which exhibits some relation to the limit problem.

In Section 22.4, we report the heuristic procedure of [EgMeSa07]. In this latter article, we addressed a more complicated problem including a variational structure, somewhat analogous to the shell problem, but simpler, concerning an equation instead of a system. It is shown that the limit problem contains in particular an elliptic Cauchy problem. This problem was handled in both a rigorous (very abstract) framework and using a heuristic procedure for exhibiting the structure of the solutions with very small ε. The reasons why the solution leaves the distribution space as ε goes to 0 are then evident. In Section 22.4 we present a simplified version of the heuristic procedure involving only the essential facts of the approximation, which are very much analogous to the method of construction of a parametrix in elliptic problems [Ta81], [EgSc97]:

• Only principal (with higher differentiation order) terms are taken into account.

• Locally, the coefficients are considered to be constant, their values being frozen at the corresponding points.

• After the Fourier transformation ($x \to \xi$), terms with small ξ are neglected in comparison with those with larger ξ (which amounts to taking into account singular parts of the solutions while neglecting smoother ones). We note that this approximation, along with the two previous ones, leads to some kind of "local Fourier transformation," which we shall use freely in the sequel.

Another important ingredient of the heuristics is a previous drastic restriction of the space where the variational problem is handled. In order to search for the minimum of energy, we only take into account functions such that the energy of the limit problem is very small. This is done using a boundary layer method within the previous approximations, i.e., for large $|\xi|$. This leads to an approximate simpler formulation of the problem for small ε, where it is apparent that the limit problem involves a smoothing operator and cannot have a solution within distribution theory.

It should prove useful to give an example of a sequence of functions converging to an analytical functional (but leaving the distribution space, then leading to a "complexification" phenomenon). It is known ([Sc50], [GeCh64]) that (direct and inverse) Fourier transformation within distribution theory is only possible for temperate distributions, not allowing functions with exponential growth at infinity. The space of (direct or inverse) Fourier transforms of general distributions is denoted by Z'. It is a space of analytical functionals: the corresponding test functions are analytical, rapidly decreasing functions, forming the space Z.

Let us consider the (nontemperate) distribution (or function) $\hat{u}(\xi) = \cosh(\xi)$. The sequence

$$\hat{u}^{\lambda}(\xi) = \begin{cases} \cosh(\xi) \text{ if } |\xi| < \lambda, \\ 0 \text{ otherwise} \end{cases}$$

converges to $\hat{u}$ in the distribution sense as λ goes to infinity. The inverse Fourier transforms $u^\lambda(x)$ converge in Z' to the analytical functional $u(x)$. The functions $\hat{u}^\lambda(\xi)$ are tempered and their inverse Fourier transforms are easily computed by hand. It appears that for large λ

$$u^\lambda(x) \approx \frac{e^\lambda}{2\pi}\frac{1}{1+x^2}(\cos(\lambda x) + x\sin(\lambda x)).$$

It is then apparent that $u^\lambda(x)$ consists of an "almost periodic" function with period tending to zero along with $1/\lambda$, multiplied by an "envelope" defined by $\frac{1}{1+x^2}$ and by the factor $\frac{e^\lambda}{2\pi}$. Moreover, note that the amplitude is exponentially large with respect to the inverse of the period. It is then apparent that the limit is an "extremely singular" function as the "graph" fills the entire plane. Moreover, it is clear (and may be rigorously proved [EgMeSa07]) that the sequence u^λ leaves the distribution space everywhere, not only in the vicinity of $x = 0$ as is suggested by the formal inverse Fourier transform of $\cosh(\xi) = \Sigma_{n=0}^{+\infty}\frac{\xi^{2n}}{(2n)!}$, which is

$$u(x) = \Sigma_{n=0}^{+\infty}\frac{-i}{(2n)!}\delta^{2n}(x),$$

apparently a singularity "of order infinity" at the origin. This fact constitutes an example of the property that elements of Z' can only be tested with analytic functions (with support on the entire x-axis) so that elements of Z' do not enjoying localization properties.

The motivation for studying this kind of problem comes from shell theory, see [SaHuSa97], [BeMiSa08]. It appears that when the middle surface is elliptic (both principal curvatures have the same sign) and is fixed by a part Γ_0 of the boundary and free by the rest Γ_1, the "limit problem" as the thickness ε tends to zero is elliptic, with boundary conditions satisfying SL on Γ_0, and boundary conditions not satisfying SL on Γ_1. Without going into details, which may be found in [MeSa06], [MeEtAl07], [EgMeSa07], and [EgMeSa09], we show numerical computations taken from [BeMiSa08] of the normal displacement for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-5}$ (Figure 22.1 left and right, respectively) when the shell is acted upon by a normal density of forces on a rectangular region of the plane of parameters. The most important feature is constituted by large oscillations near the free boundary Γ_1. It is apparent that, when passing from $\varepsilon = 10^{-3}$ to $\varepsilon = 10^{-5}$, the amplitude of the oscillations grows from 0.001 to 0.01. The singularities produced by the jump of the applied forces inside the domain are still apparent for $\varepsilon = 10^{-3}$, but not for $\varepsilon = 10^{-5}$, where only oscillations along the boundary are visible. Moreover, the number of such oscillations goes from nearly 3 for $\varepsilon = 10^{-3}$ to nearly 5 for $\varepsilon = 10^{-5}$ and is then nearly proportional to $\log(1/\varepsilon)$. We shall see that all these features agree with our theory.

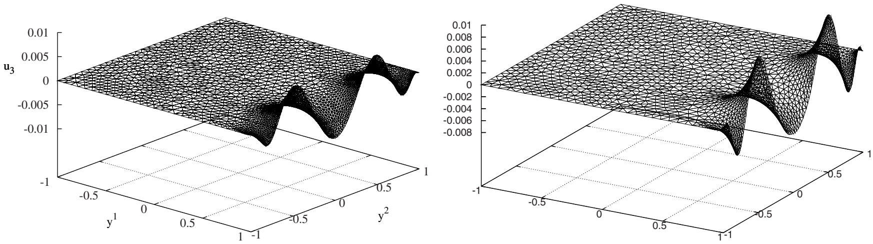

Fig. 22.1. Normal displacement for $\varepsilon = 10^{-3}$ (left) and for $\varepsilon = 10^{-5}$ (right).

22.2 The Shapiro–Lopatinskii Condition for Boundary Conditions of Elliptic Equations

In this section, we recall some properties of elliptic Partial Differential equation (PDEs) (see [AgDoNi59] and [EgSc97] for more details).

We consider a PDE of the form

$$P(x, \partial_\alpha)u = f(x),$$

where $x = (x_1, x_2)$ and $\partial_\alpha = \partial/\partial x_\alpha$, $\alpha = 1, 2$, and P is a polynomial of degree $2m$ in ∂_α. Let P_0 be the "principal part," i.e., the terms of higher order. The equation is said to be elliptic at x if the homogeneous polynomial of degree $2m$ in ξ_α:

$$P_0(x, -i\xi_\alpha) = 0 \tag{22.1}$$

has no solution $\xi = (\xi_1, \xi_2) \neq (0, 0)$ with real ξ_α. When the coefficients are real (this is the only case that we shall consider), this implies that the degree is even (this is the reason why we denoted it by $2m$). The left-hand side of (22.1) is said to be the "principal symbol;" the "symbol" is obtained in an analogous way taking the whole P instead of the principal part P_0. We note that replacing $\partial/\partial x_\alpha$ by $-i\xi_\alpha$ in P_0 amounts to formally taking the Fourier transform $x \to \xi$ for the homogeneous equation with constant coefficients obtained by discarding the lower order terms and freezing the coefficients at x. Obviously, ellipticity on a domain Ω is defined as ellipticity at any $x \in \Omega$.

It is worthwhile mentioning that ellipticity amounts to non-existence of "traveling waves" of the form

$$e^{-i\xi x} \tag{22.2}$$

for the equation obtained after discarding lower order terms and freezing coefficients. Here "traveling" amounts to "with real ξ"; note that solutions like (22.2) with non-real ξ are necessarily exponentially growing or decaying (in modulus) in some direction. Moreover, when a solution of the form (22.2) exists (with ξ either real or not), it also exists for $c\xi$ with any c. In a heuristic framework, we may suppose that $|\xi|$ is very large; this justifies discarding lower order terms (= of lower degree in $|\xi|$). In the same (heuristic) order of ideas, freezing the coefficients allows us to consider "local solutions." This amounts to multiplying the solutions by a "cutoff" function $\theta(x)$ or, equivalently, taking the convolution of the Fourier transform with $\hat{\vartheta}(\xi)$, which does not modify the behavior for large ξ. Microlocal analysis gives a rigorous sense to that heuristics. It then appears that local singularities of a solution u (associated with behavior of the Fourier transform for large $|\xi|$) cannot occur in elliptic equations unless they are controlled by the (Fourier transform of the) right-hand side f. This gives a "heuristic proof" of the classical property that local solutions of elliptic equations are rigorously associated with singularities of f.

What happens with solutions near the boundary? A local Fourier transform is no longer possible, but, after rectification of the boundary in the neighborhood of a point, we may perform a tangential Fourier transform. If, for instance, the considered part of the boundary is on the axis x_1 and the domain is on the side $x_2 > 0$, taking only higher order terms and frozen coefficients, we have solutions of the form (22.2) with real ξ_1 (coming from the Fourier transform) and non-real ξ_2. The dependence in x_2 is immediately obtained by solving an ordinary differential equation (ODE) with constant coefficients. Obviously, the solutions are exponentially growing or decreasing, for $x_2 > 0$. As the coefficients are real, there are precisely m (linearly independent) growing and m decreasing solutions (in the case of multiple roots, dependence in x_2 of the form $x_2 e^{\lambda_2}$ and analogous forms also occur). Roughly speaking, there are solutions of the form

$$\sum_k C_k e^{-i\xi_1 x_1} e^{\lambda_k x_2}$$

with real ξ_1 and $Re(\lambda) \neq 0$ (here k is running from 1 to $2m$). Boundary conditions on $x_2 = 0$ should control solutions with $Re(\lambda) < 0$, i.e., exponentially decreasing inside the domain, whereas exponentially growing ones should be controlled "by the equation in the rest of the domain and the boundary conditions on the other parts of the boundary." In other words, "good boundary conditions" should determine (within our approximation of the half-plane and frozen coefficients) the solutions of the equation of the form (22.1) with $Re(\lambda) < 0$. Obviously, the number of such boundary conditions is m. A set of m boundary conditions enjoying the above property is said to satisfy the Shapiro–Lopatinskii condition. There are several equivalent specific definitions of it. We shall mainly use the following one.

Definition 1. *Let P be elliptic at a point O of the boundary. A set of m boundary conditions $B_j(x, \partial_\alpha) = g_j(x)$, $j = 1, ..., m$ is said to satisfy the SL condition at O when, after a local change to new coordinates with origin at O and axis x_1 tangent to the boundary, taking only the higher order terms and coefficients frozen at O in the equation and the boundary conditions, the solutions of the form (22.1) with $Re(\lambda) < 0$ obtained by formal tangential Fourier transform are well defined by the boundary conditions.*

Remark 1. The preceding definition should be understood in the sense of formal solution for any given (real and nonzero) ξ_1. The SL condition is not concerned with solutions in certain spaces. It is purely algebraic, and concerns m conditions imposed to the m (decreasing with x_2) linearly independent solutions of the ODE obtained from P_0 by a formal tangential Fourier transform. This also amounts to saying that imposing the boundary conditions equal to zero, the considered solutions must vanish. In fact, the SL condition amounts to nonvanishing of a certain determinant, and as such it is generically satisfied: conditions that do not satisfy it are rarely encountered. In particular, in "well-behaved problems," when coerciveness on appropriate spaces is proved, the SL condition is not usually checked. Also note that the SL condition is independent of a change of variables, and, in most cases, the change is trivial. On the other hand, there are also definitions of the SL condition without a change of variables. Last, also note that the SL condition has nothing to do with lower order terms and the right-hand side of the boundary conditions (as ellipticity is only concerned with the principal symbol); it is merely a condition of adequation of the principal part of the boundary operators to the principal part of the equation.

Let us consider, as an exercise, examples for the Laplacian:

$$P = -\partial_1^2 - \partial_2^2. \tag{22.3}$$

The principal symbol is $\xi_1^2 + \xi_2^2$, so the equation is elliptic of order 2; thus $m = 1$. "Good boundary conditions" are in number of 1.

Let us try the boundary condition (Dirichlet)

$$u = 0. \tag{22.4}$$

Taking any point of the boundary and (x_1, x_2) with origin at that point, tangent and normal to the boundary, respectively, the equation is the same as that for the initial variables, and a formal tangential Fourier transform gives

$$(\xi_1^2 - \partial_2^2)\hat{u}(\xi_1, x_2) = 0,$$

and the solutions are

$$\hat{u}(\xi_1, x_2) = C_1(\xi_1)e^{|\xi_1|x_2} + C_2(\xi_1)e^{|\xi_1|x_2}.$$

Taking only the exponentially decreasing solutions for $x_2 > 0$ we only have

$$\hat{u}(\xi_1, x_2) = C_1(\xi_1)e^{-|\xi_1|x_2}. \tag{22.5}$$

Now, applying the "tangential Fourier transformation" to (22.4), we find that

$$\hat{u}(\xi_1, 0) = 0, \tag{22.6}$$

that is, the transform vanishes identically. Then the Dirichlet boundary condition satisfies the SL condition for the Laplacian.

The case of the Neumann boundary condition for the Laplacian

$$\frac{\partial u}{\partial n} = 0$$

is analogous. (Note also that the Fourier condition $(\frac{\partial u}{\partial n}) + au = g$ is the same, as only the higher order terms are taken in consideration.) Proceeding as before, we have, instead of (22.6):

$$\partial_2 \hat{u}(\xi_1, 0) = -|\xi_1| C_1(\xi_1) = 0,$$

which also gives $C_1(\xi_1) = 0$ and then $\hat{u} = 0$. Thus, (22.6) satisfies SL for (22.3).

In contrast, the boundary condition

$$(\partial_s - i\partial_n)u = 0, \tag{22.7}$$

where s and n denote the arc of the boundary and the normal, does not satisfy the SL condition for the Laplacian. Indeed, taking the new local axes, s and n become x_1 and x_2, and after a tangential Fourier transform

$$(-i\xi_1 - i\partial_2)\hat{u}(\xi_1, 0) = 0,$$

which applied to (22.5) becomes

$$(-i\xi_1 + i|\xi_1|)C_1(\xi_1) = 0,$$

we then see that $C_1(\xi_1)$ vanishes for negative ξ_1, but is arbitrary for positive ξ_1. In fact, the boundary condition (22.7) is "transparent" for solutions of the form (22.5) with positive ξ_1.

Remark 2. As is apparent in the last example, when the SL condition is not satisfied, there is some kind of "local nonuniqueness," where "local" recalls that only higher order terms are taken in consideration, and the coefficients are frozen at the considered point of the boundary.

The SL condition appears as some previous condition for solving elliptic problems. It is apparent that some pathology is involved at points of the boundary where it is not satisfied.

Let us mention, before closing this section, that the boundary conditions may be different on different parts of the boundary, especially on different connected components of it (when there are points of junction of the various regions, usually singularities appear at those points).

22.3 An Explicit Perturbation Problem Where the SL Condition Is Not Satisfied on a Part of the Boundary of the Limit Problem

Let Ω be the strip $(-\infty, +\infty) \times (0, 1)$ of the (x, y) plan. We denote by Γ_0 and Γ_1 the boundaries $y = 0$ and $y = 1$, respectively. We then consider the boundary value problem depending on the parameter ε:

$$\begin{cases} \triangle u^\varepsilon = 0 \text{ on } \Omega \\ u^\varepsilon = 0 \text{ on } \Gamma_0 \\ \partial_x u + (i+\varepsilon^2)\partial_y u = \varphi \text{ on } \Gamma_1 \end{cases},$$

where φ is the data of the problem. It is a given function of x, that we shall suppose sufficiently smooth, tending to 0 at infinity. We shall solve it by an $x \to \xi$ Fourier transform; it is easily seen that we also automatically have $u \to 0$ for $x \to \infty$, which may be added to the boundary conditions.

The boundary condition on Γ_0 is the Dirichlet one, which satisfies SL for the Laplacian. In contrast, the boundary condition on Γ_1 satisfies it for $\varepsilon > 0$ (this is easily checked), not at the limit $\varepsilon = 0$ (see the end of the previous section). The problem is to solve for $\varepsilon > 0$ and to study the behavior for ε going to zero.

Denoting by $\hat{}$ the $x \to \xi$ Fourier transform, $\hat{u}^\varepsilon$ is defined on the same Ω domain, but of the (ξ, y) plane. The solutions of the (transform of) equation and the boundary condition on Γ_0 are of the form

$$\hat{u}^\varepsilon(\xi, y) = \alpha(\xi)\sinh(\xi y),$$

where α denotes an unknown function to be determined with the boundary condition on Γ_1. It will prove useful to write the solution under the form

$$\hat{u}^\varepsilon(\xi, y) = \hat{\beta}^\varepsilon(\xi)\frac{\sinh(\xi y)}{\sinh(\xi)} \tag{22.8}$$

for the new unknown $\hat{\beta}^\varepsilon(\xi)$, which is the transform of the trace $u^\varepsilon(x, 0)$. Imposing the Fourier transform of the boundary condition on Γ_1, we have

$$-i\xi\hat{\beta}^\varepsilon(\xi) + (i+\varepsilon^2)\frac{\cosh(\xi)}{\sinh(\xi)}\hat{\beta}^\varepsilon(\xi)\xi = \hat{\varphi}(\xi),$$

so that

$$\hat{\beta}^\varepsilon(\xi) = \frac{\hat{\varphi}(\xi)}{-i\xi\Big(1-\coth(\xi)\Big) + \varepsilon^2\xi\coth(\xi)}. \tag{22.9}$$

In order to study this function, we should keep in mind that the expression $(1 - \coth(\xi))$ decays for $\xi \to +\infty$ as $2e^{-2\xi}$. Then, at the limit $\varepsilon = 0$ we have

$$\hat{\beta}^0(\xi) = \frac{\hat{\varphi}(\xi)}{-i\xi(1-\coth(\xi))}. \tag{22.10}$$

For $\xi \to +\infty$ this function behaves as

$$\hat{\beta}^0(\xi) \approx 2\frac{\hat{\varphi}(\xi)}{-i\xi}e^{2\xi}.$$

This shows (except for very special data φ with a very fast decaying Fourier transform) that $\hat{\beta}^0(\xi)$ is not a tempered distribution, and the inverse Fourier transform is an analytical function in $\mathcal{Z}'$. Nevertheless, for $\varepsilon > 0$, $\hat{\beta}^\varepsilon(\xi)$ is "well behaved" for $\xi \to +\infty$ as

$$\hat{\beta}^\varepsilon(\xi) \approx \frac{\hat{\varphi}(\xi)}{\xi\varepsilon^2}. \tag{22.11}$$

This specific behavior depends on that of $\frac{\hat{\varphi}}{\xi}$, so that in most cases it will be decreasing, but multiplied by the factor ε^{-2}. When $\varepsilon > 0$ (small but not 0) is fixed, $\hat{\beta}^\varepsilon(\xi)$ is approximately given by (22.10) for "finite" ξ and by (22.11) for ξ going to $+\infty$. It is easily seen that the sup in modulus of $|\hat{\beta}^\varepsilon(\xi)|$ is located in the region where both terms in the denominator of the right-hand side of (22.9) are of the same order (so that neither of them may be neglected). This gives

$$\xi = \mathcal{O}(\log(1/\varepsilon)). \tag{22.12}$$

It appears that $\hat{\beta}^\varepsilon(\xi)$ consists mainly of Fourier components which tend to infinity algebraically as ε goes to zero with ξ tending to infinity "slowly" as in (22.12). This is somewhat analogous to the example, given in the Introduction, of a sequence of functions converging to an analytical functional.

Coming back to (22.8), the main properties of the behavior of $u^\varepsilon(x,1)$ may be shown:

• The trace $u^\varepsilon(x,1) = \beta^\varepsilon(x)$ on the boundary Γ_1 which bears the "pathological boundary condition" mainly consists of large oscillations with wave length $1/\log(1/\varepsilon)$ (which tends to 0 very slowly as $\varepsilon \to 0$). The amplitude of those oscillations grows nearly as ε^{-2}. The limit $\varepsilon \to 0$ does not exist in distribution theory; it constitutes a complexification process.

• Out of the trace on Γ_1 (i.e., for $0 < y < 1$), the behavior is analogous, but of lower amplitude, which is exponentially decreasing going away from Γ_1. We recover properties of the nonuniqueness associated with the failed SL condition.

Before concluding this section, we would like to show some analogy between the previous limit problem and the Cauchy elliptic problem, which is a classical example of an ill posed problem, without a solution in general.

We consider the same domain Ω as before, but we now impose two boundary conditions on Γ_0 and no condition on Γ_1. Namely,

$$\begin{cases} \triangle v = 0 \text{ on } \Omega \\ v = \psi \text{ on } \Gamma_0 \\ \partial_y v = 0 \text{ on } \Gamma_0 \end{cases}.$$

Taking as above the $x \to \xi$ Fourier transform, it follows immediately that

$$\hat{v}(\xi, y) = \hat{\psi}(\xi)\cosh(\xi y).$$

It is apparent that the behavior for $\xi \to \infty$ is exponentially growing (except for the case when $\hat{\psi}(\xi)$ decays faster than $e^{-|\xi|}$) so that it is not tempered and the inverse Fourier transform does not exist within distribution theory.

22.4 A Model Variational Sensitive Singular Perturbation

22.4.1 Formulation of the Problem

Let Ω be a two-dimensional compact manifold with smooth (of C^∞ class) boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ of the variable $x = (x_1, x_2)$, where Γ_0 and Γ_1 are disjoint; they are one-dimensional compact smooth manifolds without boundary, then diffeomorphic to the unit circle. Let a and b be the bilinear forms given by

$$\begin{aligned} a(u,v) &= \int_\Omega \triangle u \, \triangle v \mathrm{d}x, \\ b(u,v) &= \int_\Omega \sum_{\alpha,\beta=1}^{2} \partial_{\alpha\beta} u \, \partial_{\alpha\beta} v \mathrm{d}x. \end{aligned}$$

We consider the following variational problem (which has possibly only a formal sense):

$$\begin{cases} \text{Find } u^\varepsilon \in V \text{ such that, } \forall v \in V \\ a(u^\varepsilon, v) + \varepsilon^2 b(u^\varepsilon, v) = \langle f, v \rangle, \end{cases} \tag{22.13}$$

where the space V is the "energy space" with the essential boundary conditions on Γ_0

$$V = \{v \in H^2(\Omega);\; v_{|\Gamma_0} = \frac{\partial v}{\partial n}_{|\Gamma_0} = 0\},$$

where n, t denotes the normal and tangent unit vectors to the boundary Γ with the convention that the normal vector n is inward to Ω. It is easily checked that the bilinear form b is coercive on V. Moreover, we immediately obtain the following result. For all $\varepsilon > 0$ and for all f in V', the variational problem (22.13) is of Lax–Milgram type and it is a self-adjoint problem which has a coerciveness constant larger than $c\varepsilon^2$, with $c > 0$.

The equation on Ω associated with problem (22.13) is

$$(1 + \varepsilon^2)\triangle^2 u^\varepsilon = f \text{ on } \Omega, \tag{22.14}$$

as both forms a and b give the Laplacian. As for the boundary conditions on Γ_0, they are "principal," i.e., they are included in the definition on V (defined in Section 22.4.1). As for conditions on Γ_1, they are "natural," classically obtained from the integrated terms by parts. Those coming from the form b are somewhat complicated; we shall not write them, as the problem with $\varepsilon > 0$ is classical. For $\varepsilon = 0$ these conditions (coming from form a) are $\triangle u = \frac{\partial \triangle u}{\partial n} = 0,$ on Γ_1.

As a matter of fact, the full limit boundary value problem is

$$\begin{cases} \triangle^2 u^0 = f \text{ on } \Omega \\ u = \frac{\partial u^0}{\partial n} = 0 \text{ on } \Gamma_0 \\ \triangle u^0 = 0 \text{ on } \Gamma_1 \\ -\frac{\partial}{\partial n} \triangle u^0 = 0 \text{ on } \Gamma_1. \end{cases} \tag{22.15}$$

Let us check that the boundary conditions on Γ_1 (i.e., the two last lines of (22.15)) do not satisfy the SL condition for the elliptic operator $\triangle^2$. Indeed, proceeding as in Section 22.2, by a formal tangential Fourier transform,

$$(-\xi_1^2 + \partial_2^2)^2 \hat{u} = 0,$$

which yields

$$\hat{v} = (A e^{-|\xi_1| x_2} + C x_2 e^{-|\xi_1| x_2}) \tag{22.16}$$

(as well as analogous terms with $+|\xi|$ instead of $-|\xi|$, which are not taken into account as exponentially growing inwards the domain). Here, according to SL theory, x_2 is the coordinate normal to the boundary, after taking locally tangent and normal axes (which do not modify the equation $\triangle^2$). The (tangential Fourier transform of the) boundary conditions on Γ_1 are

$$(-\xi_1^2 + \partial_2^2)\hat{u} = 0$$

and

$$\partial_2(-\xi_1^2 + \partial_2^2)\hat{u} = 0.$$

It is immediately seen that the previous solutions (22.16) with $C = 0$ and any $A \neq 0$ satisfy both conditions (note that its Laplacian vanishes everywhere, then it vanishes as well as its normal derivative on the boundary). So, the SL condition is not satisfied on Γ_1.

Before going on with our study, we note that the limit problem (22.15) implies an elliptic Cauchy problem for the auxiliary unknown

$$v^0 = \triangle u^0.$$

Indeed, system (22.15) gives in particular:

$$\begin{cases} \triangle v^0 = f \text{ on } \Omega \\ v^0 = 0 \text{ on } \Gamma_1 \\ -\frac{\partial v^0}{\partial n} = 0 \text{ on } \Gamma_1, \end{cases}$$

which is precisely the Cauchy problem for the Laplacian.

As mentioned in Section 22.3, this is a classical ill posed problem, and the solution does not exist in general. However, uniqueness of the solution holds true (by the uniqueness theorem of Holmgren and other analogous ones (see, for example, [CoHi62])).

22.4.2 A Heuristic Integral Approach

The aim of this section is the construction, in a heuristic way, of an approximate description of the solutions u^ε of the model problem in the previous section for small values of ε.

From the general theory of singular perturbations of the form (22.13), we know that our assumption,

$$a(v,v)^{1/2} \text{ defines a norm on } V, \tag{22.17}$$

is crucial. Indeed, when it is not satisfied, the problem is said to be "non-inhibited." In such a case, it has a kernel which contains non-vanishing terms, and then it is easy to establish that the asymptotic behavior of the solution u^ε of (22.13) is described by a variational problem in this kernel. This fact is not surprising when we consider the following minimization problem, which is equivalent to (22.13):

$$\begin{cases} \text{Minimize in } V, \\ a(u^\varepsilon, u^\varepsilon) + \varepsilon^2 b(u^\varepsilon, u^\varepsilon) - 2\langle f, u^\varepsilon \rangle. \end{cases} \tag{22.18}$$

Indeed, when ε goes to zero, the natural trend consists in avoiding the a-energy which occurs with the factor 1 and leaving the b-energy which has a factor ε^2.

Clearly, this is not possible when (22.17) is satisfied, since the kernel reduces to the zero function. Nevertheless, in our case, $a(v,v) = 0$ implies $\triangle v = 0$ and, as $v \in V$, the traces of v and $\frac{\partial v}{\partial n}$ vanish on Γ_0, so that (22.17) follows from the uniqueness theorem for the Cauchy problem. This uniqueness is classical, but the solution u is unstable in the sense that there can be "large u" in the V norm (or in other spaces) for "small f" in the V' norm (or in other spaces). It then appears that the same reasoning shows that for small values of ε, the solution u^ε will be precisely among elements with small $a(u^\varepsilon, u^\varepsilon)$; that is, with small $\triangle u^\varepsilon$ in L^2.

22.4.3 The Γ_0 Layer

Let us now build such functions $u^\varepsilon \in V$ with very small $\|\triangle u^\varepsilon\|_{L^2}$. The main idea is to consider functions in a larger space than the space of functions v of V such that $\triangle v = 0$ (which only contains the function $v = 0$). The functions of this bigger space will not satisfy the two boundary conditions on Γ_0 that are satisfied by any function of V. Then we shall modify it in a narrow boundary layer along Γ_0 in order to satisfy the two boundary conditions with small value of a-energy.

More precisely, let us consider the vector space

$$G^0 = \{v \in C^\infty(\overline{\Omega}),\ \triangle v = 0 \text{ on } \Omega,\ v = 0 \text{ on } \Gamma_0\}.$$

Remark 3. We observe that every function of G^0 satisfies one of the boundary conditions on Γ_0 which are satisfied by any element of V. For simplicity, we have chosen $v = 0$ on Γ_0, but we could choose the other one $\frac{\partial v}{\partial n} = 0$ on Γ_0 as well. On the other hand, the regularity assumption C^∞ is slightly arbitrary. Since we will consider the completion of G^0 with respect to some norm, this point is irrelevant.

Obviously, as the Dirichlet problem for the Laplacian on Ω is well posed in C^∞, the space G^0 is isomorphic with the space of traces on Γ_1:

$$\{w \in C^\infty(\Gamma_1)\},$$

and the isomorphism is obtained by solving the Dirichlet problem:

$$\begin{cases} \triangle \tilde{w} = 0 \text{ on } \Omega, \\ \tilde{w} = 0 \text{ on } \Gamma_0, \\ \tilde{w} = w \text{ on } \Gamma_1. \end{cases} \tag{22.19}$$

In the sequel, we shall consider either the functions $\tilde{w}$ on $\overline{\Omega}$ or their traces w on Γ_1.

In fact, the exact function u^ε is a solution of (22.14), which we are searching to describe approximately in order to define a space as small as possible (incorporating the main features of the solution) to solve the minimization problem. More precisely, according to our previous comments, we are interested in the "most singular parts" of u^ε in the sense of the part corresponding to the high frequency Fourier components. As we shall see in the sequel, it turns out that these singular parts may be obtained by modification of the functions $\tilde{w}$ on a boundary layer close to Γ_0; this layer is narrower when the considered Fourier components are of higher frequency; in fact, the layer only exists because we only consider high frequencies. This allows us to make an approximation which consists in using locally curvilinear coordinates defined by the arc of Γ_0 and the normal, and handling them as Cartesian coordinates. Clearly, this approximation is exact only on Γ_0, but is more and more precise as we approach Γ_0, i.e., as the considered frequencies grow.

Once the layer is constructed, we compute its a-energy, as well as the $\varepsilon^2 b$-energy of the (modified) $\tilde{w}$ function, in order to consider the variational problem (22.13) in the restricted space.

Let us first exhibit the local structure of the Fourier transform of $\tilde{w}$ close to Γ_0. According to our general considerations on the heuristic procedure, $\hat{w}$ may be considered (after multiplying by an appropriate cutoff function) of "small support" near a point P_0 of Γ_0. Taking local tangent and normal Cartesian coordinates y_1, y_2, we have, within our approximation,

$$\Big(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\Big)\tilde{w} = 0 \text{ on } \mathbf{R} \times (0,t), \tag{22.20}$$

for some $t > 0$. Taking the tangential Fourier transform, we obtain

$$\mathcal{F}(\tilde{w}_j)(\xi_1, y_2) = \lambda e^{|\xi_1| y_2} + \mu e^{-|\xi_1| y_2}. \tag{22.21}$$

It is worthwhile defining the local structure of $\hat{w}$ in the vicinity of Γ_0 using the "Cauchy" data $\tilde{w}$ and $\partial_2 \tilde{w}$ on Γ_0 (note that the solution of the Cauchy problem is unique, so that the Cauchy data determine the solution). As $\hat{w}$ vanishes on Γ_0, the local structure is then determined by $\partial_2 \tilde{w}$ on Γ_0. Taking the tangential Fourier transform, this gives

$$\mathcal{F}\Big(\tilde{w}_j\Big)(\xi_1, y_2) = \mathcal{F}\Big(\frac{\partial \tilde{w}_j}{\partial y_2}_{|y_2=0}\Big)\frac{\sinh(|\xi_1| y_2)}{|\xi_1|}. \tag{22.22}$$

We now proceed to the modification of $\tilde{w}$ into $\tilde{w}^a$ in a narrow boundary layer of Γ_0 in order to satisfy (always within our approximation) the equation coming from (22.14) for small ε. Using considerations similar to those leading to (22.20), this amounts to

$$\Big(\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2}\Big)^{(2)}\tilde{w}^a = 0 \text{ on } \mathbf{R}\times(0,t), \tag{22.23}$$

hence the tangential Fourier transform reads

$$\Big(-|\xi_1|^2+\frac{\partial^2}{\partial y_2^2}\Big)^{(2)}\mathcal{F}(\tilde{w}^a) = 0. \tag{22.24}$$

Consequently, $\mathcal{F}(\tilde{w}^a)$ should take the form

$$\mathcal{F}(\tilde{w}^a)(\xi_1,y_2) = (\alpha+\gamma y_2)e^{|\xi_1|y_2}+(\beta+\delta y_2)e^{-|\xi_1|y_2}.$$

The four unknown constants should be determined by imposing that $\tilde{w}^a$ and $\partial_2\tilde{w}^a$ vanish for $y_2=0$ and the "matching condition" of the layer, i.e., out of the layer, we want $\tilde{w}^a_j$ to match with the given function $\tilde{w}_j$. Since $|\xi_1| >> 1$, then $|\xi_1|y_2 >> 1$ means that $y_2 >> \frac{1}{|\xi_1|}$ (but we still impose that y_2 is small in order to be in a narrow layer of Γ_0); this is perfectly consistent, as we will only use the functions for large $|\xi_1|$. Hence, the terms with coefficients β and δ are "boundary layer terms" going to zero out of the layer (i.e., for $|y_2| >> \mathcal{O}\Big(\frac{1}{|\xi_1|}\Big)$); see perhaps [Ec79] or [Il91] for generalities on boundary layers and matching. This gives

$$\mathcal{F}\Big(\tilde{w}_j\Big)(\xi_1,y_2) = \mathcal{F}\Big(\frac{\partial\tilde{w}_j}{\partial y_2}_{|y_2=0}\Big)\big(\frac{\sinh(|\xi_1|y_2)}{|\xi_1|}-y_2e^{-|\xi_1|y_2}\big).$$

This amounts to saying that the modification of the function $\tilde{w}_j$ consists in adding to it the inverse Fourier transform of

$$\mathcal{F}\Big(\frac{\partial\tilde{w}_j}{\partial y_2}_{|y_2=0}\Big)\Big(-y_2e^{-|\xi_1|y_2}\Big).$$

Defining on Γ_0 the family (with parameter y_2) of pseudo-differential smoothing operators $\delta\sigma(\varepsilon,D_1,y_2)$ with symbol

$$\delta\sigma(\varepsilon,\xi_1,y_2) = -y_2e^{-|\xi_1|y_2}h(\varepsilon,\xi,y_2), \tag{22.25}$$

where h is an irrelevant cutoff function avoiding low frequencies that is equal to 1 for high frequencies (see [EgMeSa07] for details), we see that the modification of the function $\tilde{w}$:

$$\delta\tilde{w} = \tilde{w}^a - \tilde{w}$$

is precisely the action of $\delta\sigma(\varepsilon,D_1,y_2)$ on $\frac{\partial\tilde{w}_j}{\partial y_2}(y_1,0)$:

$$\delta\tilde{w} = \delta\sigma(\varepsilon, D_1, y_2)\frac{\partial \tilde{w}_j}{\partial y_2}(y_1, 0).$$

Let us now compute the leading terms of the a-energy of the modified function $\tilde{w}^a$.

Let $\tilde{v}$ and $\tilde{w}$ be two elements in G^0 and $\tilde{v}^a, \tilde{w}^a$ the corresponding elements modified in the boundary layer. As the given $\tilde{v}$ and $\tilde{w}$ are harmonic in Ω, the a-form is only concerned with the modification terms $\delta\tilde{v}$ and $\delta\tilde{w}$. Then, within our approximation, we have

$$a(\tilde{v}^a, \tilde{w}^a) = \int_{\Gamma_0} dy_1 \int_0^{+\infty} \triangle(\delta\tilde{v})\overline{\triangle(\delta\tilde{w})}\mathrm{d}y_2.$$

To compute this expression, we first write $\tilde{v}$ and $\tilde{w}$ as a sum of terms with "small support" (by multiplying by a partition of unity): $\tilde{v} = \Sigma_j \tilde{v}_j$ and $\tilde{w} = \Sigma_j \tilde{w}_j$. Then, within our approximation, the integral is on the half-plane $\mathbf{R} \times (0, +\infty)$ of the variables y_1, y_2. Taking the tangential Fourier transform and using the Parseval–Plancherel theorem, we have

$$a(\tilde{v}^a, \tilde{w}^a) = \Sigma_{j,k} \int_{-\infty}^{+\infty} \mathrm{d}\xi_1 \int_0^{+\infty} \Big(\frac{\mathrm{d}^2}{\mathrm{d}y_2^2} - \xi_1^2\Big)\delta\sigma(\varepsilon, \xi, y_2)\mathcal{F}\big(\frac{\partial \tilde{v}_j}{\partial y_2}_{|y_2=0}\big)$$
$$\times\overline{\Big(\frac{\mathrm{d}^2}{\mathrm{d}y_2^2} - \xi_1^2\Big)\delta\sigma(\varepsilon, \xi, y_2)\mathcal{F}\big(\frac{\partial \tilde{w}_k}{\partial y_2}_{|y_2=0}\big)}\mathrm{d}y_2.$$

Hence, from (22.25) and integrating in y_2, this yields

$$a(\tilde{v}^a, \tilde{w}^a) = \Sigma_{j,k} \int_{-\infty}^{+\infty} 2|\xi_1| \frac{\partial \tilde{w}_{1,j}}{\partial y_2}_{|y_2=0} \overline{\frac{\partial \tilde{w}_{2,k}}{\partial y_2}_{|y_2=0}} h^2(\varepsilon, \xi, y_2)\mathrm{d}\xi_1. \tag{22.26}$$

Expression (22.26) only depends on the traces $\frac{\partial \tilde{v}_j}{\partial y_2}_{|y_2=0}(y_1)$ and $\frac{\partial \tilde{w}_k}{\partial y_2}_{|y_2=0}(y_1)$, which are functions defined on Γ_0.

We now simplify this last expression using a sesquilinear form involving pseudo-differential operators.

Indeed, denoting by $P(\frac{\partial}{\partial y_1})$ the pseudo-differential operator with symbol

$$P(\xi_1) = (2|\xi_1|)^{1/2} h(\varepsilon, \xi, y_2),$$

and summing over j and k, we obtain

$$a(\tilde{v}^a, \tilde{w}^a) = \int_{\Gamma_0} P(\frac{\partial}{\partial s})\frac{\partial \tilde{v}}{\partial n}_{|\Gamma_0} \overline{P(\frac{\partial}{\partial s})\frac{\partial \tilde{w}}{\partial n}_{|\Gamma_0}} \mathrm{d}s. \tag{22.27}$$

22.4.4 Influence of the Perturbation Term $\varepsilon^2 b$

We now consider the minimization problem (22.18) on G^0 instead of on V. Obviously, the a-energy should be computed using formula (22.27). This modified problem should involve the a-energy and the $\varepsilon^2 b$-energy. A natural space for handling it should be the completion G of G^0 with the norm

$$\|v\|_G^2 = \int_{\Gamma_0} \left| P(\frac{\partial}{\partial s}) \frac{\partial v}{\partial n}_{|\Gamma_0} \right|^2 \mathrm{d}s + b(v,v).$$

It is easily seen that G is the space of the harmonic functions of $H^2(\Omega)$ vanishing on Γ_0; according to (22.19) it may be identified with the space of traces $H^{3/2}(\Gamma_1)$.

It will prove useful to write another (asymptotically equivalent for large $|\xi_1|$) definition of this problem. Indeed, the elements $\tilde{w}$ of G^0 (and then of G) may be identified (by solving the problem (22.19)) with their traces w on Γ_1. Moreover, as the functions $\tilde{w}$ are harmonic, we may exhibit their local behavior in the vicinity of any point $x_0 \in \Gamma_1$. Proceeding as in (22.20), (22.21), and taking only the decreasing exponential towards the domain (this is the classical approximation for the construction of a parametrix), we have

$$\mathcal{F}(\tilde{w})(\xi_1, y_2) = \mathcal{F}(w)(\xi_1) e^{-|\xi_1| y_2}, \tag{22.28}$$

where y_1, y_2 are the tangent and the normal (inward to the domain) vectors. Then, it is apparent that the b-energy is concentrated in a layer close to Γ_1 and we may compute it in an analogous way to the calculus that was done for the a-energy (22.27). Indeed, using the Parseval–Plancherel theorem and within our approximation, we have

$$\begin{aligned} b(\tilde{w},\tilde{w}) &= \int_{-\infty}^{+\infty} \mathrm{d}y_1 \int_0^{+\infty} \sum_{\alpha,\beta} |\partial_{\alpha\beta}\tilde{w}|^2 \mathrm{d}y_2 \\ &= \int_{-\infty}^{+\infty} \mathrm{d}\xi_1 \int_0^{+\infty} \Big(\xi_1^4 |\mathcal{F}(\tilde{w})|^2 + 2\xi_1^2 |\mathcal{F}(\frac{\partial \tilde{w}}{\partial y_2})|^2 + |\mathcal{F}(\frac{\partial^2 \tilde{w}}{\partial y_2^2})|^2 \Big) \mathrm{d}y_2. \end{aligned}$$

Hence, recalling (22.28) and integrating over y_2, we get

$$b(\tilde{w},\tilde{w}) = 2 \int_{-\infty}^{+\infty} |\xi_1|^3 |\mathcal{F}(w)|^2 \mathrm{d}\xi_1.$$

Then, defining the pseudo-differential operator $Q(\frac{\partial}{\partial s})$ of order 3/2 with principal symbol

$$\sqrt{2}|\xi_1|^{3/2},$$

or equivalently as previously,

$$\sqrt{2}(1+|\xi_1|^2)^{3/4},$$

we have (always within our approximation)

$$b(\tilde{v},\tilde{w}) = \int_{\Gamma_1} Q(\frac{\partial}{\partial s}) v \, \overline{Q(\frac{\partial}{\partial s}) w} \mathrm{d}s. \tag{22.29}$$

We observe that the operator Q is only concerned with the trace on Γ_1, so that we may either write $\tilde{v}$, $\tilde{w}$ or v, w in (22.29).

The formal asymptotic problem becomes

$$\left\{ \begin{array}{l} \text{Find } \tilde{v}^\varepsilon \in G \text{ such that } \forall \tilde{w} \in G \\ \int_{\Gamma_0} P(\frac{\partial \tilde{v}^\varepsilon}{\partial n}) \overline{P(\frac{\partial \tilde{w}}{\partial n})} \mathrm{d}s + \varepsilon^2 \int_{\Gamma_1} Q(\tilde{v}^\varepsilon) \, \overline{Q(\tilde{w})} \mathrm{d}s = \langle f, w \rangle. \end{array} \right. \tag{22.30}$$

22.4.5 The Formal Asymptotics and Its Sensitive Behavior

In order to exhibit more clearly the unusual character of the problem, we shall now write (22.30) under another equivalent form involving only the traces on Γ_1. Coming back to (22.19), let us define $\mathcal{R}_0$ as follows. For a given $w \in C^\infty(\Gamma_1)$ we solve (22.19) and we take the trace of $\frac{\partial \tilde{w}}{\partial n}$ on Γ_0, and then

$$\frac{\partial \tilde{w}}{\partial n}_{|\Gamma_0} = \mathcal{R}_0 w. \tag{22.31}$$

Using the regularity properties of the solution of (22.19), it follows that $\mathcal{R}_0 w$ is in $C^\infty(\Gamma_0)$. In fact, R_0 is a smoothing operator, sending any distribution into a C^∞ function. Then, (22.30) may be written as a problem for the traces on Γ_1:

$$\begin{cases} \text{Find } v^\varepsilon \in H^{3/2}(\Gamma_1) \text{ such that } \forall w \in H^{3/2}(\Gamma_1) \\ \int_{\Gamma_0} P(\frac{\partial}{\partial s})\mathcal{R}_0 v^\varepsilon \overline{P(\frac{\partial}{\partial s})\mathcal{R}_0 w} \mathrm{d}s + \varepsilon^2 \int_{\Gamma_1} Q(\frac{\partial}{\partial s}) v^\varepsilon \, \overline{Q(\frac{\partial}{\partial s}) w} \mathrm{d}s = \int_\Omega F \tilde{w} \mathrm{d}x, \end{cases} \tag{22.32}$$

where the configuration space is obviously $H^{3/2}(\Gamma_1)$. The left-hand side with $\varepsilon > 0$ is continuous and coercive. We then define the new operators

$$\begin{aligned} \mathcal{A} &= \mathcal{R}_0^* P^* P \mathcal{R}_0 \ \in \mathcal{L}(H^s(\Gamma_1), H^r(\Gamma_0)), \forall s, r \in \mathbf{R}, \\ \mathcal{B} &= Q^* Q \ \in \mathcal{L}(H^{3/2}(\Gamma_1), H^{-3/2}(\Gamma_1)), \end{aligned}$$

where $\mathcal{R}_0^*$ is the adjoint of $\mathcal{R}_0$ (which is also smoothing), and (22.32) becomes

$$\left(\mathcal{A} + \varepsilon^2 \mathcal{B}\right) v^\varepsilon = F, \text{ in } H^{-3/2}(\Gamma_1).$$

Obviously, $\mathcal{B}$ is an elliptic pseudo-differential operator of order 3, whereas $\mathcal{A}$ is a smoothing (non-local) operator.

This problem is somewhat simpler than the initial one (as on a manifold of dimension 1), showing the interest of the formal asymptotics. It enters in a class of sensitive problems addressed in [EgMeSa07] Section 2. It is apparent that the limit problem (for $\varepsilon = 0$) has no solution in the distribution space for any F not contained in $\mathcal{C}^\infty$. Indeed, on the compact manifold Γ_0, any distribution is in some $H^{-m}(\Gamma_0)$ space, which is sent into $\mathcal{C}^\infty$ by the smoothing operator $\mathcal{A}$.

Remark 4. The drastically non-local character of the smoothing operator $\mathcal{A}$ follows from the fact that it involves $\mathcal{R}_0$ and $\mathcal{R}_0^*$ (see (22.31)). This is the reason why the problem may be reduced to another one on the traces on Γ_1. The possibility of that reduction is a consequence of our approximation, where the configuration space is formed by harmonic functions.

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23

Regularity of the Green Potential for the Laplacian with Robin Boundary Condition

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23.1 Introduction and Statement of the Main Results

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let ν be the outward unit normal for Ω. For $\lambda \in [0, \infty]$, the Poisson problem for the Laplacian $\Delta = \sum_{i=1}^{n} \partial_i^2$ in Ω with homogeneous Robin boundary condition reads

$$\begin{cases} \Delta u = f \text{ in } \Omega, \\ \partial_\nu u + \lambda \mathrm{Tr}\, u = 0 \text{ on } \partial\Omega, \end{cases} \tag{23.1}$$

where $\partial_\nu u$ denotes the normal derivative of u on $\partial\Omega$ and Tr stands for the boundary trace operator. In the case when $\lambda = \infty$, the boundary condition in (23.1) should be understood as $\mathrm{Tr}\, u = 0$ on $\partial\Omega$. The solution operator to (23.1) (i.e., the assignment $f \mapsto u$) is naturally expressed as

$$\mathbb{G}_\lambda f(x) := \int_\Omega G_\lambda(x, y) f(y)\, dy, \quad x \in \Omega, \tag{23.2}$$

where G_λ is the Green function for the Robin Laplacian. That is, for each $x \in \Omega$, G_λ satisfies

$$\begin{cases} \Delta_y G_\lambda(x, y) = \delta_x(y), \ y \in \Omega, \\ \partial_{\nu(y)} G_\lambda(x, y) + \lambda\, G_\lambda(x, y) = 0, \quad y \in \partial\Omega, \end{cases} \tag{23.3}$$

where δ_x is the Dirac distribution with mass at x. The scope of this chapter is to investigate mapping properties of the operator $\nabla \mathbb{G}_\lambda$ when acting on $L^1(\Omega)$, the Lebesgue space of integrable functions in Ω. In this regard, weak-L^p spaces over Ω, which we denote by $L^{p,\infty}(\Omega)$, play an important role (for a precise definition see Section 23.2). The following theorem summarizes the regularity results for G_λ and $\mathbb{G}_\lambda$ proved in this chapter.

C. Constanda and M.E. Pérez (eds.), *Integral Methods in Science and Engineering, Volume 1: Analytic Methods*, DOI 10.1007/978-0-8176-4899-2_23,

Theorem 1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and fix $\lambda \in [0,\infty]$. Then*

$$\nabla[G_\lambda(x,\cdot)] \in L^{\frac{n}{n-1},\infty}(\Omega) \text{ uniformly in } x \in \Omega. \tag{23.4}$$

In particular,

$$\nabla\mathbb{G}_\lambda : L^1(\Omega) \to L^{\frac{n}{n-1},\infty}(\Omega) \text{ is a bounded operator.} \tag{23.5}$$

A number of results in the spirit of Theorem 1 are known for the Green function and the Green potential for the Laplacian on a bounded Lipschitz domain when the boundary condition is of Dirichlet or Neumann type. The fact that the gradient of the Dirichlet Green potential $\mathbb{G}_D$ maps boundedly $L^1(\Omega)$ into $L^{\frac{n}{n-1},\infty}(\Omega)$ was proved by B. Dahlberg (see [Da79]). His proof relies on the use of the maximum principle, and it cannot be used to handle a Neumann boundary condition. This obstacle was overcome in [Mi08], where a new approach was devised to prove that when Ω is a bounded Lipschitz domain, the Neumann Green function satisfies $\nabla[G_N(x,\cdot)] \in L^{\frac{n}{n-1},\infty}(\Omega)$, uniformly for $x \in \Omega$, and that $\nabla\mathbb{G}_N$, the gradient of the corresponding Neumann Green potential, maps $L^1(\Omega)$ into $L^{\frac{n}{n-1},\infty}(\Omega)$ boundedly. A key ingredient in [Mi08] is establishing the membership of the normal and tangential derivatives to the boundary of Ω of the free space fundamental solution for the Laplacian to a weak Hardy space, which is done there by employing Clifford algebras. This Clifford algebra approach cannot be readily adapted to the setting of elliptic systems. To handle the case of a second order, constant coefficient, elliptic system in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, a new technique was developed in [Mi07] for the proof of the membership to a weak Hardy space of the co-normal and tangential derivatives to $\partial\Omega$ of the corresponding fundamental (matrix) solution. With this in hand, it was then proved in [Mi07] that when $n = 3$, $\nabla[G_D(x,\cdot)]$ and $\nabla[G_N(x,\cdot)]$ belong to $L^{\frac{3}{2},\infty}(\Omega)$, uniformly for $x \in \Omega$, and that $\nabla\mathbb{G}_D$ and $\nabla\mathbb{G}_N$ map $L^1(\Omega)$ boundedly into $L^{\frac{3}{2},\infty}(\Omega)$. The topic of this chapter is a natural continuation of this line of research since we address here the more general case of the Robin boundary condition (which contains as particular cases the Dirichlet and Neumann boundary conditions). The proof of Theorem 1 is contained in Section 23.3. Various definitions, notation, and some preliminary results are collected in Section 23.2.

23.2 Preliminaries

Let (X,μ) be a measure space and for a measurable function $f : X \to \mathbb{R}$ set

$$m(\lambda, f) := \mu(\{x \in X : |f(x)| > \lambda\}), \qquad \forall\, \lambda > 0, \tag{23.6}$$

and define the non-increasing rearrangement of f as

$$f^*(t) := \inf\{\lambda > 0 : m(\lambda, f) \le t\}, \qquad t > 0. \tag{23.7}$$

In particular, $m(\lambda, f) = m(\lambda, f^*)$ for every $\lambda > 0$. If $0 < p \leq \infty$, $0 < q \leq \infty$, consider the Lorentz scale (see, e.g., [BeLo76])

$$L^{p,q}(X) := \Big\{ f : X \to \mathbb{R} \text{ measurable} : \, t^{1/p} f^*(t) \in L^q((0,\infty), \tfrac{dt}{t}) \Big\}, \tag{23.8}$$

equipped with the quasi-norm

$$\|f\|_{L^{p,q}(X)} := \|t^{1/p} f^*(t)\|_{L^q((0,\infty), \frac{dt}{t})}. \tag{23.9}$$

Note that the scale of Lorentz spaces contains Lebesgue spaces

$$L^{p,p}(X) = L^p(X), \quad 0 < p \leq \infty. \tag{23.10}$$

Also, an equivalent quasi-norm for the case when $q = \infty$ and $0 < p \leq \infty$, corresponding to weak-L^p spaces, is

$$\|f\|_{L^{p,\infty}(X)} \approx \sup \{\lambda (m(\lambda, f))^{\frac{1}{p}} : \, \lambda > 0\}. \tag{23.11}$$

For further reference, we note that when X is σ-finite and non-atomic,

$$\Big(L^{p,q}(X)\Big)^* = L^{p',\infty}(X) \quad \text{for } 1 < p < \infty, \; 0 < q \leq 1, \text{ and } \frac{1}{p} + \frac{1}{p'} = 1. \tag{23.12}$$

Recall that a function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is called Lipschitz provided there exists a constant $M > 0$ such that $\|\nabla\varphi\|_{L^\infty(\mathbb{R}^{n-1})} < M$. An unbounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ is the upper graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, i.e.,

$$\Omega = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \, x_n > \varphi(x')\}.$$

A domain $\Omega \subset \mathbb{R}^n$ is called a bounded Lipschitz domain provided its boundary $\partial\Omega$ can be covered by finitely many balls $\{B(x_i, R_i)\}_{1 \leq i \leq N}$, $x_i \in \partial\Omega$, $R_i > 0$, with the property that for each i there exists an unbounded Lipschitz domain Ω_i (considered in a system of coordinates which is a rotation and a translation of the original one) such that $\Omega \cap B(x_i, R_i) = \Omega_i \cap B(x_i, R_i)$, $1 \leq i \leq N$. See, e.g., the definition and comments on p. 189 in Stein's book [St70].

Later on, it will be useful for us to note that, for each Lipschitz domain $\Omega \subset \mathbb{R}^n$ and each $\alpha > 0$,

$$|x - \cdot|^{-\alpha} \in L^{\frac{n}{\alpha},\infty}(\Omega) \;\text{ and }\; |x - \cdot|^{-\alpha} \in L^{\frac{n-1}{\alpha},\infty}(\partial\Omega) \;\text{ uniformly in } x \in \mathbb{R}^n. \tag{23.13}$$

For $1 \leq p \leq \infty$ we denote by $L^p(\Omega)$ the Lebesgue measurable functions which are pth power integrable on Ω. It is well known that for each Lipschitz domain $\Omega \subset \mathbb{R}^n$ there is a canonical surface measure $d\sigma$, with respect to which the outward unit normal, ν, is well defined at almost every boundary point. As such, the Lebesgue space of measurable functions which are pth power integrable with respect to $d\sigma$ on $\partial\Omega$ is meaningful, and we denote it by $L^p(\partial\Omega)$. Moreover, the L^p-based Sobolev space of order one on $\partial\Omega$ will be denoted by $L^p_1(\partial\Omega)$.

For a fixed function ψ in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi \neq 0$, and for $t > 0$, $x \in \mathbb{R}^n$, we let $\psi_t(x) := t^{-n}\psi\left(\frac{x}{t}\right)$. Then, for $0 < p < \infty$, the local Hardy spaces are defined as

$$h^p(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \sup_{0<t<1} |(f * \psi_t)| \in L^p(\mathbb{R}^n)\}, \tag{23.14}$$

equipped with the quasi-norm

$$\|f\|_{h^p(\mathbb{R}^n)} := \Big\| \sup_{0<t<1} |(f * \psi_t)| \Big\|_{L^p(\mathbb{R}^n)}. \tag{23.15}$$

Hereafter $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of tempered distributions in $\mathbb{R}^n$. Weak local Hardy spaces are defined in a similar manner to (23.14). Concretely, for $0 < p < \infty$,

$$h^{p,\infty}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \sup_{0<t<1} |(f * \psi_t)| \in L^{p,\infty}(\mathbb{R}^n)\}, \tag{23.16}$$

equipped with the quasi-norm

$$\|f\|_{h^{p,\infty}(\mathbb{R}^n)} := \Big\| \sup_{0<t<1} |(f * \psi_t)| \Big\|_{L^{p,\infty}(\mathbb{R}^n)}. \tag{23.17}$$

Parenthetically, let us point out that different choices of the Schwartz function ψ give equivalent quasi-norms in (23.15), (23.17).

In the case when $\Omega \subset \mathbb{R}^n$ is the domain lying above the graph of the Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, for $\frac{n-1}{n} < p < \infty$, we define local Hardy spaces on $\partial\Omega$ by using an appropriate change of coordinates, i.e.,

$$f \in h^p(\partial\Omega) \overset{def}{\Longleftrightarrow} f(x', \varphi(x'))\sqrt{1 + |\nabla\varphi(x')|^2} \in h^p(\mathbb{R}^{n-1}), \tag{23.18}$$

and

$$\|f\|_{h^p(\partial\Omega)} := \Big\| f(x', \varphi(x'))\sqrt{1 + |\nabla\varphi(x')|^2} \Big\|_{h^p(\mathbb{R}^{n-1})}. \tag{23.19}$$

For $\frac{n-1}{n} < p < \infty$ we also set

$$f \in h^{p,\infty}(\partial\Omega) \overset{def}{\Longleftrightarrow} f(x', \varphi(x'))\sqrt{1 + |\nabla\varphi(x')|^2} \in h^{p,\infty}(\mathbb{R}^{n-1}), \tag{23.20}$$

and

$$\|f\|_{h^{p,\infty}(\partial\Omega)} := \Big\| f(x', \varphi(x'))\sqrt{1 + |\nabla\varphi(x')|^2} \Big\|_{h^{p,\infty}(\mathbb{R}^{n-1})}. \tag{23.21}$$

In the case when $\Omega \subset \mathbb{R}^n$ is a fixed, arbitrary bounded Lipschitz domain, local Hardy spaces $h^p(\partial\Omega)$, $\frac{n-1}{n} < p < \infty$, can be defined by lifting their Euclidean counterpart via a standard localization procedure (involving a smooth partition of unity subordinate to a covering of $\partial\Omega$ with coordinate balls) and a change of variables of the form (23.18). We remark that, for each $1 < p < \infty$, $h^p(\partial\Omega) = L^p(\partial\Omega)$ holds. Weak local Hardy spaces $h^{p,\infty}(\partial\Omega)$,

$\frac{n-1}{n} < p < \infty$, can also be introduced in a similar fashion. When $1 < p < \infty$, $h^{p,\infty}(\partial\Omega) = L^{p,\infty}(\partial\Omega)$.

We next recall a few interpolation results that will be used in Section 23.3 First, the real interpolation method gives that for $\frac{n-1}{n} < p_0, p_1 < \infty$, $0 < \theta < 1$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, there holds

$$(h^{p_0}(\partial\Omega), h^{p_1}(\partial\Omega))_{\theta,\infty} = h^{p,\infty}(\partial\Omega). \tag{23.22}$$

Second, the Lorentz spaces $L^{p,q}(\Omega)$ arise naturally via real interpolation between Lebesgue spaces over Ω. More precisely, for $0 < p_0 < p_1 \leq \infty$,

$$(L^{p_0}(\Omega), L^{p_1}(\Omega))_{\theta,q} = L^{p,q}(\Omega), \tag{23.23}$$

if $p_0 < q \leq \infty$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and $0 < \theta < 1$.

The duality result

$$\left(L^{n,1}(\Omega)\right)^* = L^{\frac{n}{n-1},\infty}(\Omega), \tag{23.24}$$

which is a consequence of (23.12), will be useful in the proof of Theorem 1.

Given a Lipschitz domain $\Omega \subset \mathbb{R}^n$, for some fixed, sufficiently large $\kappa = \kappa(\partial\Omega) > 0$, we set

$$\gamma(x) := \{y \in \Omega : |x-y| \leq \kappa \operatorname{dist}(y, \partial\Omega)\}, \qquad x \in \partial\Omega. \tag{23.25}$$

Then if u is defined in Ω, $\mathcal{N}(u)$, the non-tangential maximal function of u, is defined at boundary points by

$$\mathcal{N}(u)(x) := \sup\{|u(y)| : y \in \gamma(x)\}, \qquad x \in \partial\Omega. \tag{23.26}$$

Next, we introduce layer potentials associated to the Laplacian. To do so, we first recall the fundamental solution for Δ in $\mathbb{R}^n$,

$$\Gamma(x) = \begin{cases} \frac{1}{\omega_{n-1}(2-n)} \frac{1}{|x|^{n-2}}, & n \geq 3, \\ \\ \frac{1}{2\pi} \log|x|, & n = 2, \end{cases} \tag{23.27}$$

where ω_{n-1} is the surface measure of the unit sphere in $\mathbb{R}^n$. Then, the single-layer potential operator, acting on an arbitrary function $\psi : \partial\Omega \to \mathbb{R}$, is defined according to

$$\mathcal{S}\psi(x) := \int_{\partial\Omega} \Gamma(x-y)\psi(y)\,d\sigma(y), \quad \text{for } x \in \Omega. \tag{23.28}$$

The normal derivative and the boundary trace of (23.28) are, respectively, given by

$$\partial_\nu \mathcal{S}\psi = \left(-\tfrac{1}{2}I + K^*\right)\psi \quad \text{on} \quad \partial\Omega, \tag{23.29}$$

$$\operatorname{Tr}\left[\mathcal{S}\psi\right] = S\psi \quad \text{on} \quad \partial\Omega, \tag{23.30}$$

where K^* is the principal-value operator

$$K^*\psi(x) := p.v.\int_{\partial\Omega} \partial_{\nu(x)}[\Gamma(y-x)]\psi(y)\,d\sigma(y), \quad \text{for a.e. } x \in \partial\Omega, \tag{23.31}$$

and

$$S\psi(x) := \int_{\partial\Omega} \Gamma(x-y)\psi(y)\,d\sigma(y), \quad \text{for } \ x \in \partial\Omega. \tag{23.32}$$

Finally, before we proceed with the proof of Theorem 1, we record the following useful result (for a reference, see the discussion in [DaKe87]).

Proposition 1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then, for each $\frac{n-1}{n} < p < \infty$, the following holds:*

$$K^* : h^p(\partial\Omega) \longrightarrow h^p(\partial\Omega) \quad \textit{is a bounded operator}, \tag{23.33}$$

and

$$S : h^p(\partial\Omega) \longrightarrow h^p(\partial\Omega) \quad \textit{is a compact operator.} \tag{23.34}$$

Strictly speaking, (23.34) is not explicitly stated in [DaKe87] but follows from the mapping properties of the single-layer operator proved there and known (compact) embedding results.

23.3 Proof of Theorem 1

The strategy for the proof of (23.4) is to express the Green function $G_\lambda(x, y)$ as the difference between the fundamental solution for the Laplacian in the entire space and a harmonic correction. The harmonic correction is, in turn, written as an appropriate layer potential operator acting on a linear combination between the trace of the fundamental solution for the Laplacian and its normal derivative on the boundary (see (23.38) below). Then, it suffices to show that this linear combination belongs to the Hardy space $h^{1,\infty}(\partial\Omega)$ and that the gradient of the aforementioned layer potential operator maps $h^{1,\infty}(\partial\Omega)$ into the desired weak Lebesgue space. To execute this plan, we start by proving the following invertibility result.

Theorem 2. *Suppose that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Then, there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that for each $1 - \varepsilon < p < 2 + \varepsilon$ the operator*

$$-\tfrac{1}{2}I + K^* + \lambda S : h^p(\partial\Omega) \longrightarrow h^p(\partial\Omega) \tag{23.35}$$

is an isomorphism. Moreover, the operator

$$-\tfrac{1}{2}I + K^* + \lambda S : h^{1,\infty}(\partial\Omega) \longrightarrow h^{1,\infty}(\partial\Omega) \quad \textit{is an isomorphism.} \tag{23.36}$$

Proof. Combining the results in in [DaKe87] and [Br95], it follows that there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that $-\frac{1}{2}I + K^*$ is an isomorphism from the space $\{f \in h^p(\partial\Omega) : \int_{\partial\Omega} f\, d\sigma = 0\}$ into itself, for $1 - \varepsilon < p < 2 + \varepsilon$. Thus, $-\frac{1}{2}I + K^*$ is a Fredholm operator with index zero from $h^p(\partial\Omega)$ into $h^p(\partial\Omega)$, for all $1 - \varepsilon < p < 2 + \varepsilon$. In addition, using (23.34), we can conclude that the operator (23.35) is Fredholm with index zero.

Next we claim that (23.35) is one to one if $p = 2$. To see the latter, let $f \in L^2(\partial\Omega)$ be such that $(-\frac{1}{2}I + K^* + \lambda S)f = 0$ and set $u := \mathcal{S}f$. Then $\Delta u = 0$ in Ω, $\mathcal{N}(u) \in L^2(\partial\Omega)$, and $\partial_\nu u + \lambda u = 0$ on $\partial\Omega$. Using this information, we can further integrate by parts to obtain

$$\begin{aligned}\int_\Omega |\nabla u|^2 dx &= -\int_\Omega u\Delta u\, dx + \int_{\partial\Omega} u(\partial_\nu u)\, d\sigma(x) \\ &= -\lambda \int_{\partial\Omega} |u|^2 d\sigma(x) \leq 0. \end{aligned} \tag{23.37}$$

From (23.37) we clearly have that $\nabla u = 0$ in Ω, and since $\partial_\nu u + \lambda u = 0$ on $\partial\Omega$, it follows that $u = 0$ on $\partial\Omega$; hence, in fact $u = 0$ in Ω. Since $\mathcal{S}f = 0$ in Ω, taking the boundary trace and using (23.30) we obtain that $Sf = 0$ on $\partial\Omega$. In turn, because $S : L^2(\partial\Omega) \to L^2_1(\partial\Omega)$ is an isomorphism (see [Ve84]), this implies $f = 0$ as desired. This completes the proof of the claim that (23.35) is one to one if $p = 2$.

Since we have already seen that (23.35) is Fredholm with index zero, we can now infer that (23.35) is an isomorphism if $p = 2$, and furthermore, by a perturbation argument, that (23.35) is an isomorphism for $2 - \varepsilon < p < 2 + \varepsilon$. On the other hand, $L^2(\partial\Omega) \hookrightarrow h^p(\partial\Omega)$ densely for $p < 2$. Consequently, the operator (23.35) has dense range for $p < 2$. Being also Fredholm with index zero when $1 - \varepsilon < p < 2$, it follows that (23.35) is an isomorphism if $1 - \varepsilon < p < 2$. This completes the proof of the fact that (23.35) is an isomorphism for $1 - \varepsilon < p < 2 + \varepsilon$. The fact that the operator in (23.36) is an isomorphism now follows by real interpolation and (23.22). This finishes the proof of Theorem 2.

Proof (of Theorem 1). Recall the discussion at the beginning of this section and start by claiming that, for each $x, y \in \Omega$, $x \neq y$, the Green function G_λ as in (23.3) has the form

$$\begin{aligned} G_\lambda(x,y) &= \Gamma(x-y) \\ &\quad -\mathcal{S}\Big[\Big(-\tfrac{1}{2}I + K^* + \lambda S\Big)^{-1}\Big(\partial_\nu \Gamma(x-\cdot) + \lambda\Gamma(x-\cdot)\Big)\Big](y). \end{aligned} \tag{23.38}$$

Note that, once we prove that the right-hand side of (23.38) is meaningful, by combining the properties of Γ with (23.29) and (23.30), it is not hard to see that G_λ as in (23.38) is a solution of (23.3), and thus the claim will follow. In fact, we will prove that the gradient in the variable y of the right-hand side of (23.38) belongs to $L^{\frac{n}{n-1},\infty}(\Omega)$, uniformly in $x \in \Omega$. To this end, based on (23.27) and (23.13), first observe that $\nabla\Gamma(x-\cdot) \in L^{\frac{n}{n-1},\infty}(\Omega)$, uniformly in $x \in \Omega$ and $\Gamma(x-\cdot) \in L^{\frac{n-1}{n-2},\infty}(\partial\Omega)$ uniformly in $x \in \Omega$. Next, we use the embedding $L^{\frac{n-1}{n-2},\infty}(\partial\Omega) \hookrightarrow h^{1,\infty}(\partial\Omega)$ (see [RuSi96]) to conclude that $\Gamma(x-\cdot) \in h^{1,\infty}(\partial\Omega)$ uniformly in $x \in \Omega$. Since it was proved in [Mi08] that $\partial_\nu\Gamma(x-\cdot) \in h^{1,\infty}(\partial\Omega)$ uniformly in x, we can now infer that $\partial_\nu\Gamma(x-\cdot) + \lambda\Gamma(x-\cdot) \in h^{1,\infty}(\partial\Omega)$ uniformly in $x \in \Omega$. As such, recalling Theorem 2, we obtain that

$$\Big(-\tfrac{1}{2}I + K^* + \lambda S\Big)^{-1}\Big(\partial_\nu\Gamma(x-\cdot) + \lambda\Gamma(x-\cdot)\Big) \in h^{1,\infty}(\partial\Omega) \qquad (23.39)$$

uniformly in $x \in \Omega$. Furthermore, since we have that the operator

$$\nabla\mathcal{S} : h^{1,\infty}(\partial\Omega) \longrightarrow L^{\frac{n}{n-1},\infty}(\Omega), \qquad (23.40)$$

is bounded (see [Mi08], page 3792 for a proof), we can conclude that the gradient in the variable y of the right-hand side of (23.38) belongs to $L^{\frac{n}{n-1},\infty}(\Omega)$ uniformly in $x \in \Omega$. Hence, the claim is proved, and in the process we have also proved (23.4).

To see why the operator in (23.5) is bounded, we recall (23.24) to further conclude that $\nabla_y G_\lambda(x,y) \in \Big(L^{n,1}(\Omega)\Big)^*$ as a function in y, uniformly for $x \in \Omega$. Since $G_\lambda(x,y) = G_\lambda(y,x)$, it follows from (23.2) that

$$(\nabla\mathbb{G}_\lambda)^* : L^{n,1}(\Omega) \longrightarrow L^\infty(\Omega)$$

is a bounded operator and, consequently,

$$\nabla\mathbb{G}_\lambda : \Big(L^\infty(\Omega)\Big)^* \longrightarrow \Big(L^{n,1}(\Omega)\Big)^* = L^{\frac{n}{n-1},\infty}(\Omega) \qquad (23.41)$$

is also bounded. In particular, since $L^1(\Omega) \hookrightarrow \Big(L^\infty(\Omega)\Big)^*$, we also have that the operator (23.5) is bounded. This completes the proof of Theorem 1.

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24

On the Dirichlet and Regularity Problems for the Bi-Laplacian in Lipschitz Domains

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24.1 Preliminaries and Statement of Main Result

Recall that a Lipschitz domain is a domain whose boundary is locally given by graphs of Lipschitz functions. The formulation of, respectively, the Dirichlet and regularity problems for the Laplacian in a Lipschitz domain $\Omega \subset \mathbb{R}^n$ is

$$(D_\Delta)_p \begin{cases} \Delta u = 0 \quad \text{in } \Omega, \\ \mathcal{N}u \in L^p(\partial\Omega), \\ u\big|_{\partial\Omega} = f \in L^p(\partial\Omega), \end{cases} \qquad (R_\Delta)_p \begin{cases} \Delta u = 0 \quad \text{in } \Omega, \\ \mathcal{N}(\nabla u) \in L^p(\partial\Omega), \\ u\big|_{\partial\Omega} = f \in L^p_1(\partial\Omega). \end{cases} \tag{24.1}$$

A few clarifications are in order here. First, the nontangential maximal operator of a given function u in Ω is defined by

$$(\mathcal{N}u)(X) := \sup\{|u(Y)| : Y \in \Gamma(X)\}, \qquad X \in \partial\Omega, \tag{24.2}$$

where, for some fixed parameter $\kappa > 0$, the nontangential approach region $\Gamma(X)$ with vertex at $X \in \partial\Omega$ is defined as

$$\Gamma(X) := \{Y \in \Omega : |X - Y| \leq (1+\kappa)\,\mathrm{dist}\,(Y, \partial\Omega)\}. \tag{24.3}$$

Second, the nontangential boundary trace of a function u defined in Ω is taken to be

$$u\Big|_{\partial\Omega}(X) := \lim_{\substack{Y \to X \\ Y \in \Gamma(X)}} u(Y), \qquad X \in \partial\Omega, \tag{24.4}$$

whenever meaningful. Also, $L^p(\partial\Omega)$, $1 < p < \infty$, is the usual Lebesgue scale of measurable, pth power integrable functions with respect to the surface measure σ on $\partial\Omega$. Going further, the Sobolev space $L^p_1(\partial\Omega)$, $1 < p < \infty$, then consists of functions f for which

$$\|f\|_{L^p_1(\partial\Omega)} := \|f\|_{L^p(\partial\Omega)} + \sum_{1 \leq j,k \leq n} \|\partial_{\tau_{jk}} f\|_{L^p(\partial\Omega)} < \infty, \tag{24.5}$$

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where, with $\nu = (\nu_1, ..., \nu_n)$ denoting the outward unit normal to $\partial\Omega$,

$$\partial_{\tau_{jk}} := \nu_j \partial_k - \nu_k \partial_j, \quad 1 \leq j, k \leq n. \tag{24.6}$$

Finally, we wish to point out that the statement that $(D_\Delta)_p$ is well posed indicates that this problem has a unique solution (for any datum f), which satisfies $\|\mathcal{N}u\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}$, uniformly in f. Analogously, saying that $(R_\Delta)_p$ is well posed amounts to the fact that this problem has a unique solution (for any datum f), which satisfies $\|\mathcal{N}(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p_1(\partial\Omega)}$, uniformly in f.

Moving on, we now consider the Dirichlet and regularity problems for the bi-Laplacian, respectively,

$$(D_{\Delta^2})_p \begin{cases} \Delta^2 u = 0 \quad \text{in } \Omega, \\ \mathcal{N}(\nabla u) \in L^p(\partial\Omega), \\ u\big|_{\partial\Omega} = f \in L^p_1(\partial\Omega), \\ \partial_\nu u = g \in L^p(\partial\Omega), \end{cases} \qquad (R_{\Delta^2})_p \begin{cases} \Delta^2 u = 0 \quad \text{in } \Omega, \\ \mathcal{N}(\nabla\nabla u) \in L^p(\partial\Omega), \\ u\big|_{\partial\Omega} = f_0 \in L^p_1(\partial\Omega), \\ (\partial_j u)\big|_{\partial\Omega} = f_j \in L^p_1(\partial\Omega), \\ \text{for each } 1 \leq j \leq n. \end{cases} \tag{24.7}$$

Here and elsewhere, ∂_ν denotes the normal derivative. Once again, we make similar conventions (as in the case of the Laplacian) regarding the sense in which the well-posedness of these two problems should be understood.

The main result of this chapter is the following theorem, relating the well-posedness of the regularity problems both for Δ and Δ^2 for some $p \in (1, \infty)$ to the well-posedness of the Dirichlet problem for Δ^2 for the Hölder conjugate exponent of p. For a related result for second order operators, see [Sh07].

Theorem 1. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and that $1 < p, p' < \infty$ are such that $1/p + 1/p' = 1$. Then*

$$(R_\Delta)_{p'} \quad \textit{and} \quad (R_{\Delta^2})_p \quad \textit{well-posed} \Longrightarrow (D_{\Delta^2})_{p'} \quad \textit{well-posed.} \tag{24.8}$$

In the proof of this result we shall use the Neumann boundary conditions introduced by G. Verchota in [Ve05]. Specifically, for $\theta \in \mathbb{R}$ set

$$\begin{aligned} K_\theta(u) &:= \partial_\nu(\Delta u) + \tfrac{1}{2(1+2\theta+n\theta^2)} \sum_{i,j=1}^{n} \partial_{\tau_{ij}} \big(\partial_\nu \partial_{\tau_{ij}} u\big), \\ M_\theta(u) &:= \tfrac{2\theta+n\theta^2}{1+2\theta+n\theta^2}\, \Delta u + \tfrac{1}{1+2\theta+n\theta^2}\, \partial_\nu^2 u. \end{aligned} \tag{24.9}$$

Lemma 1. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and that $(R_\Delta)_{p'}$ is well posed for some $p \in (1, \infty)$ (where $1/p + 1/p' = 1$). Then*

$$\begin{aligned} &\Delta^2 u = 0 \ \textit{in } \Omega \ \textit{and } \mathcal{N}(\nabla\nabla u) \in L^p(\partial\Omega) \\ &\qquad\qquad \Longrightarrow (-K_\theta(u), M_\theta(u)) \in L^p_{-1}(\partial\Omega) \oplus L^p(\partial\Omega), \end{aligned} \tag{24.10}$$

where $L^p_{-1}(\partial\Omega) := \Big(L^{p'}_1(\partial\Omega)\Big)^*$, plus a natural estimate.

Proof. The membership in $L^p(\partial\Omega)$ of $M_\theta(u)$ is a consequence of the fact that $\mathcal{N}(\nabla\nabla u) \in L^p(\partial\Omega)$, plus a Fatou-type theorem which asserts that for any such biharmonic function u, the nontangential pointwise trace $(\partial_j\partial_k u)|_{\partial\Omega}$ exists a.e. on $\partial\Omega$ for every $j,k \in \{1,...,n\}$. The same type of reasoning shows that the second term in (24.9) belongs to $L^p_{-1}(\partial\Omega)$. Finally, it remains to show that $\partial_\nu(\Delta u) \in L^p_{-1}(\partial\Omega)$. To this end, if $v := \Delta u$ in Ω, it follows that $\partial_\nu(\Delta u) = \partial_\nu v$ and

$$\Delta v = 0 \text{ in } \Omega, \quad \mathcal{N}(v) \in L^p(\partial\Omega). \tag{24.11}$$

Thus, we are left with showing that, in general,

$$(R_\Delta)_{p'} \quad \text{well-posed, and } v \text{ as in (24.11)} \\ \implies \partial_\nu v \in L^p_{-1}(\partial\Omega), \text{ plus an estimate.} \tag{24.12}$$

To justify (24.12), fix an arbitrary $f \in L^{p'}_1(\partial\Omega)$ and let w solve the regularity problem with this boundary datum. Green's formula then allows us to define $\partial_\nu v$ as a functional in $L^p_{-1}(\partial\Omega) = \left(L^{p'}_1(\partial\Omega)\right)^*$ by setting

$$\Big\langle \partial_\nu v\,,\, f \Big\rangle := \int_{\partial\Omega} v\,(\partial_\nu w)\,d\sigma. \tag{24.13}$$

Since $\|\partial_\nu w\|_{L^{p'}(\partial\Omega)} \le \|\mathcal{N}(\nabla w)\|_{L^{p'}(\partial\Omega)} \le C\|f\|_{L^{p'}_1(\partial\Omega)}$, it follows that indeed $\partial_\nu v \in L^p_{-1}(\partial\Omega)$ and $\|\partial_\nu v\|_{L^p_{-1}(\partial\Omega)} \le C\|v\|_{L^p(\partial\Omega)} \le C\|\mathcal{N}v\|_{L^p(\partial\Omega)}$. From this, the desired conclusion readily follows.

For the bi-Laplacian Δ^2, any bilinear form of the type

$$\mathcal{B}_\theta(u,v) := \sum_{i,j=1}^n \frac{1}{1+2\theta+n\theta^2} \\ \times \int_\Omega [(\partial_i\partial_j + \theta\delta_{ij}\Delta)u](X)[(\partial_i\partial_j + \theta\delta_{ij}\Delta)v](X)\,dX, \tag{24.14}$$

where $\theta \in \mathbb{R}$ is arbitrary, satisfies

$$\mathcal{B}_\theta(u,v) = \int_\Omega (\Delta^2 u)(X)\,v(X)\,dX, \quad \forall\, u,v \in C^\infty_c(\Omega). \tag{24.15}$$

Successive integrations by parts give

$$\mathcal{B}_\theta(u,v) = \int_{\partial\Omega} \Big[M_\theta(u)\partial_\nu v - K_\theta(u)v\Big]\,d\sigma \text{ if } \Delta^2 u = \Delta^2 v = 0 \text{ in } \Omega. \tag{24.16}$$

Let $B(X)$, $X \in \mathbb{R}^n \setminus \{0\}$, be the canonical fundamental solution for Δ^2 in $\mathbb{R}^n$. For each $\dot{f} = (f_0, f_1, \dots, f_n)$, the double layer for Δ^2 is then defined as

$$(\dot{\mathcal{D}}\dot{f})(X) := I + II + III, \tag{24.17}$$

where, for each $X \in \Omega$, we have set

$$
\begin{aligned}
I &:= \int_{\partial\Omega} \partial_{\nu(Y)}[(\Delta B)(X-Y)] f_0(Y)\, d\sigma(Y),\\
II &:= \int_{\partial\Omega} [(\Delta B)(X-Y)] \sum_{k=1}^{n} \nu_k(Y) f_k(Y)\, d\sigma(Y),\\
III &:= \tfrac{1}{1+2\theta+n\theta^2} \int_{\partial\Omega} \sum_{j,k=1}^{n} \sum_{i,\ell=1}^{n} \partial_{\tau_{jk}(Y)}[(\partial_k B)(X-Y)]\\
&\qquad \times \Big(\nu_i(Y)(\partial_{\tau_{ij}(Y)} f_0)(Y) - \nu_j(Y)\nu_\ell(Y) f_\ell(Y)\Big)\, d\sigma(Y).
\end{aligned}
\tag{24.18}
$$

Also, for any functional Λ acting on families of functions $(f_0, f_1, \dots, f_n)$ defined on $\partial\Omega$, the single-layer potential for the bi-Laplacian is given by

$$
(\dot{\mathcal{S}}\Lambda)(X) := \Big\langle \Big(B(X-\cdot)\big|_{\partial\Omega}, (\nabla B)(X-\cdot)\big|_{\partial\Omega}\Big), \Lambda \Big\rangle. \tag{24.19}
$$

These operators have many remarkable properties (cf. [MiMi08]). Among these, we wish to single out the fact that

$$
\Delta^2 \dot{\mathcal{D}} = \Delta^2 \dot{\mathcal{S}} = 0 \quad \text{in} \quad \Omega, \tag{24.20}
$$

and that the following Green representation formula holds for any biharmonic function u in Ω (which behaves reasonably near the boundary):

$$
u = \dot{\mathcal{D}}\Big(u\big|_{\partial\Omega}, (\nabla u)\big|_{\partial\Omega}\Big) - \dot{\mathcal{S}}\Big(-K_\theta(u), \nu_1 M_\theta(u), \dots, \nu_n M_\theta(u)\Big) \text{ in } \Omega. \tag{24.21}
$$

In addition, whenever $1 < p, p' < \infty$, $1/p + 1/p' = 1$, the estimates

$$
\begin{aligned}
&\|\mathcal{N}(\nabla \dot{\mathcal{D}} \dot{f})\|_{L_p(\partial\Omega)} \le C \|\dot{f}\|_{\dot{L}^p_{1,0}(\partial\Omega)},\\
&\|\mathcal{N}(\nabla\nabla \dot{\mathcal{D}} \dot{f})\|_{L_p(\partial\Omega)} \le C \|\dot{f}\|_{\dot{L}^p_{1,1}(\partial\Omega)},
\end{aligned}
\tag{24.22}
$$

and

$$
\begin{aligned}
&\|\mathcal{N}(\nabla\nabla \dot{\mathcal{S}} \Lambda)\|_{L^{p'}(\partial\Omega)} \le C \|\Lambda\|_{\left(\dot{L}^p_{1,0}(\partial\Omega)\right)^*},\\
&\|\mathcal{N}(\nabla \dot{\mathcal{S}} \Lambda)\|_{L^{p'}(\partial\Omega)} \le C \|\Lambda\|_{\left(\dot{L}^p_{1,1}(\partial\Omega)\right)^*}
\end{aligned}
\tag{24.23}
$$

are valid for some $C = C(\Omega, p) > 0$, uniformly in $\dot{f}$ and Λ. Above, we have used the following notation:

$$
\begin{aligned}
\dot{L}^p_{1,0}(\partial\Omega) := \Big\{ \dot{f} = (f_0, f_1, ..., f_n) \in L^p_1(\partial\Omega) \oplus L^p(\partial\Omega) \oplus \cdots \oplus L^p(\partial\Omega) :&\\
\partial_{\tau_{jk}} f_0 = \nu_j f_k - \nu_k f_j,\ 1 \le j, k \le n \Big\},&
\end{aligned}
\tag{24.24}
$$

and

$$\dot{L}^p_{1,1}(\partial\Omega) := \Big\{\dot{f} = (f_0, f_1, ..., f_n) \in L^p_1(\partial\Omega) \oplus \cdots \oplus L^p_1(\partial\Omega) : \\ \partial_{\tau_{jk}} f_0 = \nu_j f_k - \nu_k f_j,\ 1 \le j,k \le n\Big\}, \quad (24.25)$$

both equipped with natural norms.

24.2 Proof of the Main Result

This section is concerned with the proof of Theorem 1. This requires a number of preliminary results, which we now begin to address. First, we shall need the fact that the "packing" map

$$\begin{aligned} &\psi : \dot{L}^{p'}_{1,0}(\partial\Omega) \longrightarrow L^{p'}_1(\partial\Omega) \oplus L^{p'}(\partial\Omega),\ 1 < p' < \infty, \\ &\psi(\dot{f}) := \Big(f_0, \textstyle\sum_{j=1}^n \nu_j f_j\Big),\ \text{if } \dot{f} = (f_0, f_1, \dots, f_n), \end{aligned} \quad (24.26)$$

is an isomorphism, whose inverse and dual are, respectively, given by

$$\begin{aligned} &\psi^{-1} : L^{p'}_1(\partial\Omega) \oplus L^{p'}(\partial\Omega) \longrightarrow \dot{L}^{p'}_{1,0}(\partial\Omega) \\ &\psi^{-1}(F, h) = \dot{f} := \Big(F, (\nu_j h + \textstyle\sum_{k=1}^n \nu_k \partial_{\tau_{kj}} F)_{1\le j\le n}\Big), \end{aligned} \quad (24.27)$$

and, with $1/p + 1/p' = 1$,

$$\psi^* : L^p_{-1}(\partial\Omega) \oplus L^p(\partial\Omega) \to \Big(\dot{L}^{p'}_{1,0}(\partial\Omega)\Big)^*,\ \psi^*(f, g) = \Big(f, (\nu_j g)_{1\le j\le n}\Big). \quad (24.28)$$

Also,

$$\begin{aligned} &(\psi^{-1})^* = (\psi^*)^{-1} : \Big(\dot{L}^{p'}_{1,0}(\partial\Omega)\Big)^* \longrightarrow L^p_{-1}(\partial\Omega) \oplus L^p(\partial\Omega) \\ &(\psi^{-1})^* \Lambda = \Big(h_0 - \textstyle\sum_{j,k=1}^n \partial_{\tau_{kj}}(\nu_k h_j)\,,\ \sum_{j=1}^n \nu_j h_j\Big), \end{aligned} \quad (24.29)$$

if the functional $\Lambda \in \Big(\dot{L}^{p'}_{1,0}(\partial\Omega)\Big)^*$ acts as the boundary integral pairing against the $(n+1)$-tuple $(h_0, h_1, ..., h_n) \in L^p(\partial\Omega) \oplus \cdots \oplus L^p(\partial\Omega)$. In particular, it can be checked that for any reasonable function u in Ω,

$$\psi\Big(u|_{\partial\Omega}, (\nabla u)\big|_{\partial\Omega}\Big) = (u, \partial_\nu u) \text{ and } \psi^{-1}(u, \partial_\nu u) = \Big(u|_{\partial\Omega}, (\nabla u)\big|_{\partial\Omega}\Big). \quad (24.30)$$

Next, consider the regularity-to-Neumann map

$$\Lambda_{RN}(\dot{f}) := (-K_\theta(u)\,,\ M_\theta(u))\,, \quad (24.31)$$

where u solves $(R_{\Delta^2})_p$ for the boundary datum $\dot{f} \in \dot{L}^p_{1,1}(\partial\Omega)$. Observe that, in concert with (24.28), Green's integral representation formula gives

$$u = \Big(\dot{\mathcal{D}} - \dot{\mathcal{S}} \circ (\psi^* \circ \Lambda_{RN})\Big)\Big(u\big|_{\partial\Omega}, (\nabla u)|_{\partial\Omega}\Big) \quad \text{in} \quad \Omega, \tag{24.32}$$

for any biharmonic function u in Ω satisfying $\mathcal{N}(\nabla\nabla u) \in L^p(\partial\Omega)$. Thus, by also taking (24.30) into account, we arrive at the conclusion that

$$u = \Big(\dot{\mathcal{D}} - \dot{\mathcal{S}} \circ (\psi^* \circ \Lambda_{RN})\Big) \circ \psi^{-1}\Big(u\big|_{\partial\Omega}, \partial_\nu u\Big) \quad \text{in} \quad \Omega, \tag{24.33}$$

for any biharmonic function u in Ω with $\mathcal{N}(\nabla\nabla u) \in L^p(\partial\Omega)$. The significance of formula (24.33) is that it allows us to recover a biharmonic function u in Ω (with a reasonable behavior near the boundary) solely from the knowledge of its Dirichlet data $\Big(u\big|_{\partial\Omega}, \partial_\nu u\Big)$ on $\partial\Omega$. We shall use this as a blueprint for constructing a solution for $(D_{\Delta^2})_{p'}$. Concretely, given the problem

$$(D_{\Delta^2})_{p'} \begin{cases} \Delta^2 u = 0 \quad \text{in } \Omega, \\ \mathcal{N}(\nabla u) \in L^{p'}(\partial\Omega), \\ u\big|_{\partial\Omega} = f \in L_1^{p'}(\partial\Omega), \\ \partial_\nu u = g \in L^{p'}(\partial\Omega), \end{cases} \tag{24.34}$$

consider the function

$$u := \Big(\dot{\mathcal{D}} - \dot{\mathcal{S}} \circ (\psi^* \circ \Lambda_{RN})\Big) \circ \psi^{-1}\big(f, g\big) \quad \text{in} \quad \Omega. \tag{24.35}$$

By (24.20), we have that $\Delta^2 u = 0$ in Ω, and we now wish to show that $\mathcal{N}(\nabla u) \in L^{p'}(\partial\Omega)$. To this end, let us observe first that Lemma 1 and the fact that $(R_{\Delta^2})_p$ is well posed imply that

$$\Lambda_{RN} : \dot{L}^p_{1,1}(\partial\Omega) \longrightarrow L^p_{-1}(\partial\Omega) \oplus L^p(\partial\Omega) \tag{24.36}$$

is a well defined linear, and bounded operator. Thus, by (24.28), so is

$$\psi^* \circ \Lambda_{RN} : \dot{L}^p_{1,1}(\partial\Omega) \longrightarrow \Big(\dot{L}^{p'}_{1,0}(\partial\Omega)\Big)^*. \tag{24.37}$$

On the other hand, from the solvability of $(R_{\Delta^2})_2$ (cf. [PiVe95], [Ve05]), by letting u, v solve $(R_{\Delta^2})_2$ with data $\dot{f}, \dot{g}$ and using (24.16) plus the fact that the bilinear form in (24.14) is symmetric, it follows that

$$\Big\langle \Lambda_{RN}(\dot{f}) \,,\, \psi(\dot{g}) \Big\rangle = \Big\langle \psi(\dot{f}) \,,\, \Lambda_{RN}(\dot{g}) \Big\rangle \ \text{ for every } \dot{f}, \dot{g} \in \dot{L}^2_{1,1}(\partial\Omega), \tag{24.38}$$

where $\langle \cdot, \cdot \rangle$ is the integral pairing on $\partial\Omega$. Hence, $\psi^* \circ \Lambda_{RN}$ is formally self-adjoint which, given that (24.37) is bounded, allows us to conclude that

$$\psi^* \circ \Lambda_{RN} : \dot{L}^{p'}_{1,0}(\partial\Omega) \longrightarrow \Big(\dot{L}^p_{1,1}(\partial\Omega)\Big)^* \tag{24.39}$$

is also going to be a well defined, linear, and bounded operator. As a consequence of this and cf. (24.27), (24.22), and (24.23), we may therefore conclude that, for u as in (24.35), we have $\mathcal{N}(\nabla u) \in L^{p'}(\partial\Omega)$ and

$$\|\mathcal{N}(\nabla u)\|_{L^{p'}(\partial\Omega)} \leq C\Big(\|f\|_{L_1^{p'}(\partial\Omega)} + \|g\|_{L^{p'}(\partial\Omega)}\Big). \tag{24.40}$$

This estimate is also useful in the verification of the boundary conditions $u|_{\partial\Omega} = f$, $\partial_\nu u = g$, a task to which we now turn. Indeed, from to (24.40) (used together with (24.22)–(24.23)), it follows that the assignment

$$L_1^{p'}(\partial\Omega) \oplus L^{p'}(\partial\Omega) \ni (f,g) \mapsto \Big(u\big|_{\partial\Omega}, \partial_\nu u\Big) \in L_1^{p'}(\partial\Omega) \oplus L^{p'}(\partial\Omega) \tag{24.41}$$

is well defined, linear, and bounded (if u is as in (24.35)). Our goal is to prove that this is the identity operator on the space $L_1^{p'}(\partial\Omega) \oplus L^{p'}(\partial\Omega)$. Given the boundedness of (24.41), it therefore suffices to establish that (24.41) acts as the identity on a dense subspace $\mathcal{V}$ of $L_1^{p'}(\partial\Omega) \oplus L^{p'}(\partial\Omega)$. However, we know this to be the case for $\mathcal{V} := \psi\Big(\dot{L}^p_{1,1}(\partial\Omega)\Big) \cap \Big(L_1^{p'}(\partial\Omega) \oplus L^{p'}(\partial\Omega)\Big)$, from (24.32) and the well-posedness of $(R_{\Delta^2})_p$ ("reverse engineering").

In summary, the above reasoning shows that the Dirichlet problem $(D_{\Delta^2})_{p'}$ (see (24.34)) has a solution which satisfies (24.40). To finish showing that this problem is well posed, it remains to establish the uniqueness of such a solution. To get started, we first construct a suitable Green function for the bi-Laplacian. Concretely, for each $X, Y \in \Omega$, $X \neq Y$, we set

$$G(X,Y) := B(X-Y) - w^X(Y), \tag{24.42}$$

where w^X solves the regularity problem

$$\begin{cases} \Delta^2 w^X = 0 \text{ in } \Omega, \quad \mathcal{N}(\nabla\nabla w^X) \in L^p(\partial\Omega), \\ w^X|_{\partial\Omega} = B(X - \cdot)|_{\partial\Omega},\ (\nabla w^X)|_{\partial\Omega} = (\nabla B)(X - \cdot)|_{\partial\Omega}. \end{cases} \tag{24.43}$$

Then for each $X \in \Omega$ we have

$$\begin{cases} \Delta_Y^2 G(X,Y) = \delta_X(Y), \\ G(X,\cdot)|_{\partial\Omega} = (\nabla.G)(X,\cdot)|_{\partial\Omega} = 0, \\ \mathcal{N}\Big(\nabla_Y^2 G(X,Y)\Big) \in L^p(\partial\Omega). \end{cases} \tag{24.44}$$

Next, for each $\varepsilon > 0$ set

$$\Omega_\varepsilon := \{X \in \Omega :\ \mathrm{dist}(X, \partial\Omega) > \varepsilon\}, \tag{24.45}$$

and pick a family of functions $\Phi_\varepsilon \in C_c^\infty(\mathbb{R}^n)$, indexed by $\varepsilon \in (0,1)$, with the property that there exist two constants $0 < C_1 < C_2 < \infty$ such that

$$\Phi_\varepsilon \equiv 1 \text{ on } \Omega_{C_2\varepsilon},\ \ \Phi_\varepsilon \equiv 0 \text{ on } \mathbb{R}^n \setminus \overline{\Omega_{C_1\varepsilon}}\,,\ \text{ and } |\partial^\alpha \Phi_\varepsilon| \leq \frac{C_\alpha}{\varepsilon^{|\alpha|}}, \tag{24.46}$$

for each multi-index α. Fix next $X \in \Omega$, and let $\varepsilon > 0$ be small enough such that $X \in \Omega_\varepsilon$. Then, if we let u be a null solution of the problem (24.34), we may write

$$u(X) = (u\Phi_\varepsilon)(X) = \int_\Omega \Delta_Y^2 G(X,Y)\Phi_\varepsilon(Y)u(Y)\,dY. \tag{24.47}$$

Integrating by parts and utilizing the support conditions on Φ_ε, we obtain

$$u(X) = \int_\Omega G(X,\cdot)\Delta^2(\Phi_\varepsilon u) = \int_\Omega \sum_{|\alpha|=|\beta|=2} G(X,\cdot)A_{\alpha\beta}\partial^{\alpha+\beta}(\Phi_\varepsilon u), \tag{24.48}$$

for some $A_{\alpha\beta} \in \mathbb{R}$. Using Leibniz's formula $\partial^{\alpha+\beta}(\Phi_\varepsilon u) = \sum_{\alpha+\beta=\gamma+\delta} C_{\gamma\delta}^{\alpha\beta}\partial^\gamma\Phi_\varepsilon\partial^\delta u$ and the fact that $\sum_{|\alpha|=|\beta|=2} A_{\alpha\beta}C_{0(\alpha+\beta)}^{\alpha\beta}\partial^{\alpha+\beta}u = \Delta^2 u = 0$ (since $C_{0(\alpha+\beta)}^{\alpha\beta} = 1$), we conclude that

$$\sum_{|\alpha|=|\beta|=2} A_{\alpha\beta}\,\partial^{\alpha+\beta}(\Phi_\varepsilon u) = \sum_{|\alpha|=|\beta|=2} A_{\alpha\beta} \sum_{\substack{\alpha+\beta=\gamma+\delta \\ \gamma\neq 0}} C_{\gamma\delta}^{\alpha\beta}(\partial^\gamma\Phi_\varepsilon)\partial^\delta u. \tag{24.49}$$

Next, split the sum on the right-hand side of (24.49) over the set of multi-indices δ of length less than or equal to 1 and the set of multi-indices δ of length ≥ 2. In the latter case, write $\delta = \mu + \theta$ with $|\mu| = 1$. Then (24.48)–(24.49) yield

$$u(X) = I_\varepsilon + II_\varepsilon, \tag{24.50}$$

where I_ε is a linear combination of terms of the form

$$\int_\Omega G(X,Y) \sum_{|\alpha|=|\beta|=2} \sum_{\substack{\alpha+\beta=\gamma+\delta \\ \gamma\neq 0, |\delta|\leq 1}} (\partial^\gamma\Phi_\varepsilon)(Y)(\partial^\delta u)(Y)\,dY, \tag{24.51}$$

and, after integrating by parts, II_ε is a linear combination of terms like

$$\int_\Omega \sum_{|\alpha|=|\beta|=2} \sum_{\substack{\alpha+\beta=\gamma+\delta \\ \gamma\neq 0, |\delta|\geq 2}} \sum_{\substack{\delta=\mu+\theta_1+\theta_2 \\ \theta_1\neq 0, |\mu|=1}} \partial_Y^{\theta_1}G(X,Y)(\partial^{\gamma+\theta_2}\Phi_\varepsilon)(Y)(\partial^\mu u)(Y)\,dY. \tag{24.52}$$

Notice that $1 \leq |\theta_1| + |\theta_2| = 3 - |\gamma| \leq 2$, as $\gamma \neq 0$. Also, using the fact that $\gamma \neq 0$, we can replace Ω by $\Omega \setminus \Omega_\varepsilon$ as the domain of integration in (24.51) and (24.52). Going further, we break up the integral over sufficiently small domains $(U_i)_{1\leq i\leq N}$, each contained in a local coordinate system where $U_i \cap \Omega$ can be regarded as the upper graph of a Lipschitz function $\phi_i : Q_i \to \mathbb{R}$, where Q_i is a cube in $\mathbb{R}^{n-1}$. Based on these and (24.46), we may estimate $|II_\varepsilon|$ by terms of the form

$$\sum_{i=1}^{N} \int_{Q_i} \int_0^{C\varepsilon} \varepsilon^{-|\gamma|-|\theta_2|} |(\nabla u)(y', t+\phi_i(y'))| \times |(\partial_Y^{\theta_1} G)(X, (y', t+\phi_i(y')))| \, dt \, dy', \tag{24.53}$$

where the multi-indices are subject to the same conditions as above. If $|\theta_1| = 2$, we keep this in the current format. If, on the other hand, $|\theta_1| = 1$, we use the Fundamental Theorem of Calculus to write that, for each i and $y' \in Q_i$,

$$\partial_Y^{\theta_1} G(X, (y', t+\phi_i(y'))) = -\int_0^t (\partial_Y^{\theta_1+e_n} G)(X, (y', r+\phi_i(y'))) \, dr. \tag{24.54}$$

This allows us to once again have a formula involving two derivatives on G on the right-hand side of (24.54). Since on the domain of integration $|t| < C\varepsilon$, we may further conclude that, in this case, for each $y' \in Q_i$, $1 \le i \le N$,

$$\begin{aligned} |\partial_Y^{\theta_1} G(X, (y', t+\phi_i(y')))| &\le \varepsilon \sup_{0<r<C\varepsilon} |(\nabla_Y^2 G)(X, (y', r+\phi_i(y')))| \\ &\le \varepsilon \mathcal{N}(\nabla_Y^2 G(X, \cdot))(y', \phi_i(y')) \in L^p(Q_i). \end{aligned} \tag{24.55}$$

Therefore, since $-|\gamma| - |\theta_2| + 1 = -1$ when $|\theta_1| = 1$, we altogether have

$$|II_\varepsilon| \le C \sum_{i=1}^{N} \int_{Q_i} \Big(\varepsilon^{-1} \int_0^{C\varepsilon} |(\nabla u)(y', t+\phi_i(y'))| \times \mathcal{N}((\nabla_Y^2 G)(X - \cdot))(y', \phi_i(y')) \, dt \Big) \, dy'. \tag{24.56}$$

We shall employ *Lebesgue's dominated convergence theorem* to show that the expression inside parentheses above converges to zero as $\varepsilon \to 0$. To this end, we first observe that

$$\Big| \varepsilon^{-1} \int_0^{C\varepsilon} |(\nabla u)(y', t+\phi_i(y'))| \, dt \Big| \le C \mathcal{N}(\nabla u)(y', \phi_i(y')) \tag{24.57}$$

and recall that, by hypothesis, $\mathcal{N}(\nabla u)(y', \phi(y')) \in L^{p'}(Q_i)$. Thus, (24.55) ensures that the uniform pointwise domination part of Lebesgue's theorem is satisfied. As for pointwise convergence to zero, we make the simple observation that if $f : (0,1) \to \mathbb{R}$ is a continuous function and $\lim_{t\to 0^+} f(t) = 0$, then $\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^\varepsilon f(t) \, dt = 0$ (as seen easily from an application of the mean value theorem). Since, by hypothesis and by (24.30), $\lim_{t\to 0^+} (\nabla u)(y', t+\phi_i(y')) = 0$ for a.e. $y' \in Q_i$, the above observation applies and shows that, pointwise a.e., the integrand in (24.56) converges to zero. Thus, the Lebesgue dominated convergence theorem gives

$$\lim_{\varepsilon \to 0} II_\varepsilon = 0. \tag{24.58}$$

Turning our attention to I_ε, for the terms in which $\delta = 0$, we use the Fundamental Theorem of Calculus (as in the previous step) to write

$$u(y', t + \phi_i(y')) = -\int_0^t (\partial_n u)(y', r + \phi_i(y'))\, dr. \tag{24.59}$$

Given that $|t| \leq C\varepsilon$, for each $y' \in Q_i$ and $1 \leq i \leq N$, we get

$$\begin{aligned} \Big|u(y', t + \phi_i(y'))\Big| &\leq C\varepsilon \sup_{0<r<C\varepsilon} \Big|(\nabla u)(y', r + \phi_i(y'))\Big| \\ &\leq C\varepsilon \mathcal{N}(\nabla u)(y', \phi_i(y')). \end{aligned} \tag{24.60}$$

Also, proceeding as in (24.54)–(24.55), we obtain

$$|G(X, (y', t + \phi_i(y')))| \leq \varepsilon^2 \mathcal{N}(\nabla^2 G(X, \cdot))(y', \phi_i(y')), \tag{24.61}$$

given that $y' \in Q_i$, $1 \leq i \leq N$. Therefore, (24.46) and (24.60)–(24.61) give that the integrand in I_ε is pointwise dominated in each Q_i by

$$\begin{aligned} &C\varepsilon \Big[\varepsilon^{-|\gamma|} \varepsilon^2 \mathcal{N}(\nabla^2 G(X, \cdot))(y', \phi_i(y'))\Big] \Big[\varepsilon^{1-|\delta|} \mathcal{N}(\nabla u)(y', \phi_i(y'))\Big] \\ &= \mathcal{N}(\nabla u)(y', \phi_i(y')) \mathcal{N}(\nabla^2 G(X, \cdot))(y', \phi_i(y')) \in L^1(Q_i). \end{aligned} \tag{24.62}$$

Hence, the uniform domination condition in Lebesgue's theorem is satisfied. Also, as before, since for $|\theta| \leq 1$, each $i \in \{1, ..., N\}$ and a.e. $y' \in Q_i$,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^{C\varepsilon} (\partial^\theta u)(y', r + \phi_i(y'))\, dr = 0, \tag{24.63}$$

Lebesgue's dominated convergence theorem applies and gives

$$\lim_{\varepsilon \to 0} I_\varepsilon = 0. \tag{24.64}$$

Finally, (24.64), (24.58), and (24.50) give that $u(X) = 0$, and in turn $u \equiv 0$ on Ω. This finishes the proof of Theorem 1.

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25

Propagation of Waves in Networks of Thin Fibers

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25.1 Introduction

This chapter contains a simplified and improved version of the results obtained by the authors earlier. Wave propagation is discussed in a network of branched thin wave guides when the thickness vanishes and the wave guides shrink to a one-dimensional graph. It is shown that asymptotically one can describe the propagating waves, the spectrum and the resolvent in terms of solutions of ordinary differential equations (ODEs) on the limiting graph. The vertices of the graph correspond to junctions of the wave guides. In order to determine the solutions of the ODEs on the graph uniquely, one needs to know the gluing conditions (GC) on the vertices of the graph.

Unlike other publications on this topic, we consider the situation when the spectral parameter is greater than the threshold, i.e., the propagation of waves is possible in cylindrical parts of the network. We show that the GC in this case can be expressed in terms of the scattering matrices related to individual junctions. The results are extended to the values of the spectral parameter below the threshold and around it.

Consider the stationary wave (Helmholtz) equation

$$H_\varepsilon u = -\varepsilon^2 \Delta u = \lambda u, \quad x \in \Omega_\varepsilon, \quad Bu = 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{25.1}$$

in a domain $\Omega_\varepsilon \subset R^d$, $d \geq 2$, with infinitely smooth boundary (for simplicity), which has the following structure: Ω_ε is a union of a finite number of cylinders $C_{j,\varepsilon}$ (which will be called channels) of lengths l_j, $1 \leq j \leq N$, with diameters of cross sections of order $O(\varepsilon)$ and domains $J_{1,\varepsilon}, \ldots, J_{M,\varepsilon}$ (which will be called junctions) connecting the channels into a network. It is assumed that the junctions have diameters of the same order $O(\varepsilon)$. The boundary condition (BC) has the form: $B = 1$ (the Dirichlet BC) or $B = \frac{\partial}{\partial n}$ (the Neumann BC) or $B = \varepsilon\frac{\partial}{\partial n} + \alpha(x)$, where n is the exterior normal and the function $\alpha \geq 0$ is real valued and does not depend on the longitudinal (parallel to the axis) coordinate on the boundary of the channels. One also can impose one type of

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BC on the lateral boundary of Ω_ε and another BC on the free ends (which are not adjacent to a junction) of the channels.

The axes of the channels form edges Γ_j, $1 \leq j \leq N$, of the limiting ($\varepsilon \to 0$) metric graph Γ. The junctions shrink to vertices of the graph Γ when $\varepsilon \to 0$. We denote the set of vertices v_j by V. Let m channels have infinite length ($m = 0$ for bounded Ω_ε). We start the counting of $C_{j,\varepsilon}$ with the infinite channels. So, $l_j = \infty$ for $1 \leq j \leq m$. See Figure 25.1.

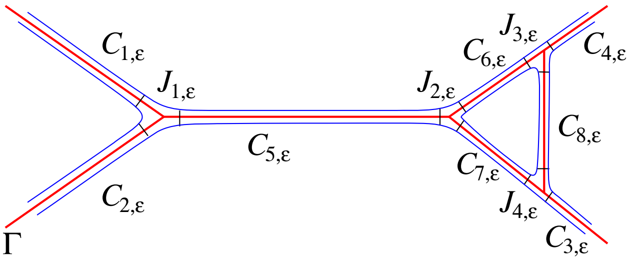

Fig. 25.1. An example of a domain Ω_ε with four junctions, four unbounded channels, and four bounded channels.

The goal of this chapter is the asymptotic analysis of the spectrum of H_ε, the resolvent $(H_\varepsilon - \lambda)^{-1}$, and solutions of the corresponding nonstationary problems for the heat and wave equations as $\varepsilon \to 0$. One can expect that H_ε is close (in some sense) to a one-dimensional operator on the limiting graph Γ with appropriate gluing conditions (GC) at the vertices $v \in V$. The ODE on Γ appears in a natural way from the following principle: the oscillating modes in the wave guides survive as $\varepsilon \to 0$ and the exponentially decaying and growing modes disappear. However, the justification of this fact is not always simple. In order to determine the solutions of ODE on Γ uniquely, one needs to know the GC on the vertices of Γ. The form of the GC in the general situation was discovered quite recently in our papers [MoVa06]–[MoVa08]. It turned out that they can be expressed in terms of scattering matrices for problems of the wave propagation through individual junctions of Ω_ε. These GC hold in all the cases: in the bulk of the spectrum $\lambda > \lambda_0$, and near the threshold $\lambda \approx \lambda_0$, for bounded and unbounded Ω_ε.

Equation (25.1) degenerates when $\varepsilon = 0$. One could omit ε^2 in (25.1). However, the problem under consideration would remain singular, since the domain Ω_ε shrinks to the graph Γ as $\varepsilon \to 0$. The presence of this coefficient in the equation is convenient, since it makes the spectrum less vulnerable to changes in ε. As we shall see, in some important cases (spider domains Ω_ε), the spectrum of the problem (25.1) does not depend on ε, and the spectrum in the same cases will be magnified by a factor of ε^{-2} if ε^2 in (25.1) is omitted. The operator $H_\varepsilon = -\varepsilon^2\Delta$ introduced in (25.1) will be considered as the operator in $L^2(\Omega_\varepsilon)$.

An important class of domains Ω_ε are the self-similar domains with only one junction and all the channels of infinite length. We shall call them *spider domains*. Thus, if Ω_ε is a spider domain, then there exists a point $\widehat{x} = x(\varepsilon)$ and an ε-independent domain Ω such that

$$\Omega_\varepsilon = \{(\widehat{x} + \varepsilon x) : x \in \Omega\}, \tag{25.2}$$

i.e., Ω_ε is the ε-contraction of $\Omega = \Omega_1$.

For simplicity, we shall assume that Ω_ε is self-similar in a neighborhood of each junction. Namely, let $J_{j(v),\varepsilon}$ be the junction which corresponds to a vertex $v \in V$ of the limiting graph Γ. Consider a junction $J_{v,\varepsilon} = J_{j(v),\varepsilon}$ and all the channels adjacent to $J_{v,\varepsilon}$. If some of these channels have finite length, we extend them to infinity. We assume that, for each $v \in V$, the resulting domain $\Omega_{v,\varepsilon}$, which consists of the junction $J_{v,\varepsilon}$ and the semi-infinite channels emanating from it, is a spider domain. We also assume here that all the channels $C_{j,\varepsilon}$ have the same cross section ω_ε. This assumption is needed only to make the results more transparent (the general case is studied in [MoVa07]). From the self-similarity assumption, it follows that ω_ε is an ε-homothety of a bounded domain $\omega \subset R^{d-1}$.

Let $\lambda_0 < \lambda_1 \le \lambda_2 \cdots$ be eigenvalues of the negative Laplacian $-\Delta_{d-1}$ in ω with the BC $B_0 u = 0$ on $\partial\omega$, where B_0 coincides with the boundary operator B on the channels, see (25.1), with $\varepsilon = 1$ in the case of the third boundary condition. Let $\{\varphi_n(y)\}$, $y \in \omega \in R^{d-1}$, be the set of corresponding orthonormal eigenfunctions. Then λ_n are eigenvalues of $-\varepsilon^2 \Delta_{d-1}$ in ω_ε, and $\{\varepsilon^{-d/2}\varphi_n(y/\varepsilon)\}$ are the corresponding eigenfunctions. In the presence of infinite channels, the spectrum of the operator H_ε consists of an absolutely continuous component which coincides with the semi-bounded interval $[\lambda_0, \infty)$ and a discrete set of eigenvalues. The eigenvalues can be located below λ_0 and can be embedded into the absolutely continuous spectrum. We will call the point $\lambda = \lambda_0$ the threshold since it is the bottom of the absolutely continuous spectrum or (and) the first point of accumulation of the eigenvalues as $\varepsilon \to 0$. Let us consider two of the simplest examples: the Dirichlet problem in a half-infinite cylinder and in a bounded cylinder of length l. In the first case, the spectrum of the negative Dirichlet Laplacian in Ω_ε is pure absolutely continuous and has multiplicity $n + 1$ on the interval $[\lambda_n, \infty)$. In the second case, the spectrum consists of the set of eigenvalues $\lambda_{n,m} = \lambda_n + \varepsilon^2 m^2/l^2$, $n \ge 0$, $m \ge 1$.

It was shown in [MoVa06]–[MoVa08] that the wave propagation governed by the operator H_ε, $\varepsilon \to 0$, as well as the asymptotic behavior of the resolvent $(H_\varepsilon - \lambda)^{-1}$ and of the eigenvalues of H_ε above λ_0 can be described in terms of the scattering solutions. While many particular cases of that problem with $\lambda = \lambda_0 + O(\varepsilon^2)$ or $\lambda < \lambda_0$ were considered (see [DeTe06]–[RuSc01]), the publications [MoVa06]–[MoVa08] were the first ones dealing with the case $\lambda \ge \lambda_0$, and the first ones where the significance of the scattering solutions for asymptotic analysis of H_ε was established. In particular, it was shown there that in both cases, $\lambda > \lambda_0$ and $\lambda \approx \lambda_0$, the GC of the operator on the limiting graph Γ will be expressed in terms of the scattering matrices of the auxiliary problems on the spider domains associated to individual junctions. A more profound analysis of the case $\lambda \approx \lambda_0$ can be found in [MoVa08].

The main goal of this chapter is to overview the results of [MoVa07]–[MoVa08] and simplify the proofs. We will mostly deal with the case of $\lambda \in (\lambda_0, \lambda_1)$, where the results and proofs are more transparent. The number of scattering solutions is the smallest in this case, and the scattering matrix is of the smallest size (compared to the case $\lambda > \lambda_1$). One of our main results is as follows.

Theorem 1. *If $\lambda_0 \leq \lambda \leq \lambda_1$, then the resolvent $(H_\varepsilon^{(1)} - \lambda)^{-1}$ can be approximated by $(H_\varepsilon^{(1)} - (\lambda - \lambda_0))^{-1}$, where $H_\varepsilon^{(1)} = -\varepsilon^2 \frac{d^2}{dt^2}$ is the operator of the second derivative defined on functions ς on the limiting graph Γ with the GC of the form*

$$i\varepsilon[I_v + T_v(\lambda)]\frac{d}{dt}\varsigma^v(0) - \sqrt{\lambda - \lambda_0}[I_v - T_v(\lambda)]\varsigma^v(0) = 0,$$

Here $T_v(\lambda)$ is the scattering matrix of the auxiliary problem on the spider domain which corresponds to the junction $J_{v,\varepsilon}$, and ς^v is the vector which consists of restrictions of the function ς (defined on Γ) onto edges adjacent to v.

To be more exact, for any compactly supported f, the following relation is valid on channels outside of the support of f with exponential accuracy:

$$(H_\varepsilon - \lambda)^{-1} f \sim [(H_\varepsilon^{(1)} - (\lambda - \lambda_0))^{-1} f_0]\varphi_0(y/\varepsilon), \quad \varepsilon \to 0, \quad f_0 = < f, \varphi_0(y/\varepsilon) > .$$

A more accurate statement of this theorem as well as some of its generalizations will be given in Section 25.5.

Note that the eigenvalues of the problem in Ω_ε are located not only below the threshold, but also above it. They depend on ε and move very fast on the λ-axis as $\varepsilon \to 0$. Thus, one cannot expect to obtain an asymptotic approximation of the resolvent $(H_\varepsilon - \lambda)^{-1}$ when $\lambda = \lambda' > \lambda_0$ is fixed and $\varepsilon \to 0$. An asymptotic approximation of the resolvent $(H_\varepsilon - \lambda)^{-1}$ as $\varepsilon \to 0$ can be valid only if an exponentially small (in ε), but depending on ε, set on the λ-axis is omitted. Another option is to fix $\lambda = \lambda' > \lambda_0$ and pass to the limit as $\varepsilon \to 0$ without ε taking values in some small set which depends on λ'.

While the condition $\lambda > \lambda_0$ is natural for the wave propagation, the properties of the heat and diffusion processes depend on the spectrum of H_ε near $\lambda = \lambda_0$. As a by-product of the simpler approach to the problem introduced below, we will get a better result concerning the asymptotic behavior of the eigenvalues of H_ε in bounded domains Ω_ε as $\varepsilon \to 0$, $\lambda = \lambda_0 + O(\varepsilon^2)$. It was shown in [MoVa07], [MoVa08] that the main terms of the eigenvalues of H_ε when $\lambda = \lambda_0 + O(\varepsilon^2)$, $\varepsilon \to 0$, coincide with the eigenvalues of the operator on the limiting graph with the GC defined by the scattering matrix at

$\lambda = \lambda_0$. An explicit description of GC at $\lambda = \lambda_0$ for arbitrary junctions (of order $O(\varepsilon)$) was also given there. Significantly later some of our results were repeated in [Gr08]. The new elements in [Gr08] are the description of the location of the eigenvalues below the threshold and more accurate asymptotics of eigenvalues near the threshold. We will show here that the approach used in [MoVa07] and [MoVa08] provides an approximation of the eigenvalues near the threshold with an exponential accuracy as well as the location of the eigenvalues below the threshold.

The plan of the chapter is as follows. The elliptic problem in Ω_ε with a fixed $\varepsilon = 1$ will be studied in the next section. In particular, the scattering solutions are defined there. The asymptotic behavior of the resolvent $(H_\varepsilon - \lambda)^{-1}$ of the spectrum and of the scattering solutions as $\varepsilon \to 0$, $\lambda > \lambda_0$, is obtained in Section 25.3 for the simplest domains with one junction (spider domains). The one-dimensional problem on the limiting graph will be studied in Section 25.4. The case of arbitrary domains Ω_ε is considered in Section 25.5. The last section is devoted to the study of the spectrum near the threshold.

25.2 Scattering Solutions and Analytic Properties of the Resolvent When ε is Fixed

We introduce Euclidean coordinates (t, y) in channels $C_{j,\varepsilon}$ chosen in such a way that the t-axis is parallel to the axis of the channel (so, t is not a time, but a space variable!), hyperplane R_y^{d-1} is orthogonal to the axis, and $C_{j,\varepsilon}$ has the following form in the new coordinates:

$$C_{j,\varepsilon} = \{(t, \varepsilon y) : 0 < t < l_j,\ y \in \omega\}.$$

If a channel $C_{j,\varepsilon}$ is bounded ($l_j < \infty$), the direction of the t axis can be chosen arbitrarily (at least for now). If a channel is unbounded, then $t = 0$ corresponds to its cross section, which is attached to the junction.

Let us recall the definition of scattering solutions for the problem (25.1) in Ω_ε when $\lambda \in (\lambda_0, \lambda_1)$. Consider the nonhomogeneous problem

$$(-\varepsilon^2 \Delta - \lambda)u = f, \quad x \in \Omega_\varepsilon; \qquad Bu = 0 \ \text{ on } \partial\Omega_\varepsilon. \tag{25.3}$$

Definition 1. *Let $f \in L^2_{com}(\Omega_\varepsilon)$ have a compact support, and $\lambda < \lambda_1$. A solution u of (25.3) is called outgoing if it has the following asymptotic behavior at infinity in each infinite channel $C_{j,\varepsilon}$, $1 \le j \le m$:*

$$u = a_j e^{i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t}\varphi_0(y/\varepsilon) + O(e^{-\frac{\alpha t}{\varepsilon}}), \quad \alpha > 0. \tag{25.4}$$

Remark 1. 1. Here and everywhere below we assume that

$$\mathrm{Im}\sqrt{\lambda - \lambda_0} \ge 0. \tag{25.5}$$

Thus, outgoing solutions decay at infinity if $\lambda < \lambda_0$.

2. Obviously, if (25.4) holds with some $\alpha > 0$, then it holds with any $\alpha < \sqrt{\lambda_1 - \lambda}$.

Definition 2. *Let $\lambda < \lambda_1$. A function $\Psi = \Psi_p^{(\varepsilon)}$, $1 \le p \le m$, is called a solution of the scattering problem in Ω_ε if*

$$(-\varepsilon^2 \Delta - \lambda)\Psi = 0, \quad x \in \Omega_\varepsilon; \qquad B\Psi = 0 \quad on\ \partial\Omega_\varepsilon, \tag{25.6}$$

and Ψ has the following asymptotic behavior in infinite channels $C_{j,\varepsilon}$, $1 \le j \le m$:

$$\Psi_p^{(\varepsilon)} = [\delta_{p,j} e^{-i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t} + t_{p,j} e^{i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t}]\varphi_0(y/\varepsilon) + O(e^{-\frac{\alpha t}{\varepsilon}}), \quad t \to \infty, \quad \alpha > 0. \tag{25.7}$$

Here $\delta_{p,j}$ is the Kronecker symbol, i.e., $\delta_{p,j} = 1$ if $p = j$, $\delta_{p,j} = 0$ if $p \ne j$.

Remark 2. If $\lambda_0 < \lambda < \lambda_1$, then the term with the coefficient $\delta_{p,j}$ in (25.7) corresponds to the incident wave (coming through the channel $C_{p,\varepsilon}$), and the terms with coefficients $t_{p,j}$ describe the transmitted waves. The transmission coefficients $t_{p,j} = t_{p,j}(\varepsilon, \lambda)$ depend on ε and λ. The matrix

$$T = [t_{p,j}] \tag{25.8}$$

is called *the scattering matrix.* Note that the scattering solution and scattering matrix are defined for all $\lambda < \lambda_1$. We assume that $\mathrm{Im}\sqrt{\lambda - \lambda_0} > 0$ when $\lambda < \lambda_0$, and the incident wave is growing (exponentially) at infinity in this case.

The outgoing and scattering solutions are defined similarly when $\lambda \in (\lambda_n, \lambda_{n+1})$ (see [MoVa07]). In this case, any outgoing solution has $n + 1$ waves in each channel propagating to infinity with the frequencies $\sqrt{\lambda - \lambda_s}/\varepsilon$, $0 \le s \le n$. There are $m(n + 1)$ scattering solutions: the incident wave may come through one of m infinite channels with one of $(n+1)$ possible frequencies. The scattering matrix has the size $m(n + 1) \times m(n + 1)$ in this case.

Standard arguments based on the Green formula provide the following statement.

Theorem 2. *When $\lambda_0 < \lambda < \lambda_1$, the scattering matrix T is unitary and symmetric ($t_{p,j} = t_{j,p}$).*

The operator $H_\varepsilon = -\varepsilon^2 \Delta$, which corresponds to the eigenvalue problem (25.1), is nonnegative, and therefore the resolvent

$$R_\lambda = (H_\varepsilon - \lambda)^{-1} : L^2(\Omega_\varepsilon) \to L^2(\Omega_\varepsilon) \tag{25.9}$$

is analytic in the complex λ-plane outside the positive semi-axis $\lambda \ge 0$. If Ω_ε is bounded (all the channels are finite), then operator R_λ is meromorphic in λ with a discrete set $\Lambda = \{\mu_{j,\varepsilon}\}$ of poles of first order at the eigenvalues $\lambda = \mu_{j,\varepsilon}$

of operator H_ε. If Ω_ε has at least one infinite channel, then the spectrum of H_ε has a more complicated structure (see Theorem 3 below). In this case, the spectrum has an absolutely continuous component $[\lambda_0, \infty)$, and the resolvent R_λ is meromorphic in $\lambda \in C\backslash[\lambda_0, \infty)$. We are going to consider the analytic extension of the operator R_λ to the absolutely continuous spectrum. One can extend R_λ analytically from above (Im$\lambda > 0$) or below, if it is considered as an operator in the following spaces (with a smaller domain and a larger range):

$$R_\lambda : L^2_{com}(\Omega_\varepsilon) \to L^2_{loc}(\Omega_\varepsilon). \qquad (25.10)$$

These extensions do not coincide on $[\lambda_0, \infty)$. To be specific, we always will consider extensions from the upper half-plane (Im$\lambda > 0$). We will call (25.10) the truncated resolvent of the operator H_ε, since it can be identified with the resolvent (25.9) multiplied from the left and right by a cut-off function.

Theorem 3. *Let Ω_ε have at least one infinite channel. Then*

(1) The spectrum of the operator H_ε consists of the absolutely continuous component $[\lambda_0, \infty)$ and, possibly, a discrete set $\{\mu_{j,\varepsilon}\}$ of nonnegative eigenvalues $\lambda = \mu_{j,\varepsilon} \geq 0$ with the only possible limiting point at infinity. The multiplicity of the absolutely continuous spectrum changes at points $\lambda = \lambda_n$, and is equal to $m(n+1)$ on the interval $(\lambda_n, \lambda_{n+1})$.

(2) The operator (25.10) admits a meromorphic extension from the upper half-plane Im$\lambda > 0$ onto $[\lambda_0, \infty)$ with the branch points at $\lambda = \lambda_n$ of the second order and poles of first order at $\lambda = \mu_{j,\varepsilon}$. In particular, if $\lambda_n \in \{\mu_{j,\varepsilon}\}$, then operator (25.10) has the form

$$R_\lambda = \frac{A(n)}{\lambda - \lambda_n} + O(\frac{1}{\sqrt{|\lambda - \lambda_n|}}), \ \lambda \to \lambda_n.$$

(3) If $f \in L^2_{com}(\Omega_\varepsilon)$, $\lambda < \lambda_1$, and λ is not a pole or the branch point ($\lambda = \lambda_0$) of the operator (25.10), then the problem (25.3), (25.4) is uniquely solvable and the outgoing solution u can be found as the $L^2_{loc}(\Omega_\varepsilon)$ limit

$$u = R_{\lambda + i0} f. \qquad (25.11)$$

(4) There exist exactly m different scattering solutions for the values of $\lambda < \lambda_1$ which are not a pole or the branch point of the operator (25.10), and the scattering solution is defined uniquely after the incident wave is chosen.

(5) The scattering matrix T is analytic in λ, when $\lambda < \lambda_1$, with a branch point of second order at $\lambda = \lambda_0$ and without real poles.

The matrix T is orthogonal if $\lambda < \lambda_0$.

Remark 3. Let $\lambda_n \notin \{\mu_{j,\varepsilon}\}$. If the homogeneous problem (25.3) with $\lambda = \lambda_n$ has a non-trivial solution u such that

$$u = a_j \varphi_n(y/\varepsilon) + O(e^{-\gamma t}), \quad x \in C_{j,\varepsilon}, \quad t \to \infty, \quad 1 \leq j \leq m, \ \gamma > 0, \ (25.12)$$

then $R_{\lambda+i0} = \frac{B(n)}{\sqrt{\lambda - \lambda_n}} + O(1), \ \ \lambda \to \lambda_n$. If such a solution u does not exist, then operator (25.10) is bounded in a neighborhood of $\lambda = \lambda_n$.

Proof (of Theorem 3). All the statements above concern the problem with a fixed value of ε and can be proved using standard elliptic theory arguments. A detailed proof can be found in [MoVa07]; a shorter version is given below.

In order to prove the part of statement (1) concerning the absolutely continuous spectrum of the operator $H = -\Delta$, we split the domain Ω_ε into pieces by introducing cuts along the bases $t = 0$ of all infinite channels. We denote the new (not connected) domain by Ω'_ε, and denote the negative Dirichlet Laplacian in Ω'_ε by H'_ε, i.e., H'_ε is obtained from H_ε by introducing additional Dirichlet boundary conditions on the cuts. Obviously, the operator H'_ε has the absolutely continuous spectrum described in statement (1) of the theorem. Since the wave operators for the couple H_ε, H'_ε exist and are complete (see [Bi62]), the operator H_ε has the same absolutely continuous spectrum as H'_ε.

The remaining part of statement (1) and statements (2) and (3) can be proved by one of the well-known equivalent approaches based on a reduction of the boundary problem (25.3) to a Fredholm equation which depends analytically on λ. For this purpose one can use a parametrix (almost inverse operator): equation (25.3) is solved separately in channels and junctions, and then the parametrix can be constructed from those local inverse operators using a partition of unity (allowing one to glue the local inverse operators); see [MoVa07]. A similar approach is based on gluing together these local inverse operators using Dirichlet-to-Neumann maps on the cuts of the channels, as introduced in the previous paragraph.

Statements (4) and (5) follow immediately from statement (3) and Theorem 2. Indeed, one can look for the solution $\Psi_p^{(\varepsilon)}$ of the scattering problem in the form

$$\Psi_p^{(\varepsilon)} = \chi e^{-i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t}\varphi_0(y/\varepsilon) + u, \tag{25.13}$$

where $\chi \in C^\infty(\Omega_\varepsilon)$, $\chi = 0$ outside $C_{p,\varepsilon}$, $\chi = 1$ in $C_{p,\varepsilon} \cap \{t > 1\}$. This reduces problem (25.6), (25.7) to problem (25.3), (25.4) for u with f supported on $C_{p,\varepsilon} \cap \{0 \le t \le 1\}$. Statement (3) of the theorem, applied to the latter problem, justifies statement (4). Function u, defined in (25.13), satisfies the homogeneous equation (25.3) in infinite channels $C_{j,\varepsilon}$, $j \ne p$, and in $C_{p,\varepsilon} \cap \{t > 1\}$, and it is meromorphic at the bottoms of these channels (at $t = 0$ for $j \ne p$, and $t = 1$ when $j = p$). Solving the problems in these channels by separation of variables, we obtain that the scattering matrix T is meromorphic in λ, when $\lambda < \lambda_1$ with a branch point of second order at $\lambda = \lambda_0$. It also follows from here that T is real valued when $\lambda < \lambda_0$. The analyticity of T and Theorem 2 imply that T is orthogonal when $\lambda < \lambda_0$. From the orthogonality $(\lambda < \lambda_0)$ and unitarity $(\lambda_0 < \lambda < \lambda_1)$ of T, it follows that T does not have poles.

25.3 Spider Domains, $\varepsilon \to 0$

Let us recall that Ω_ε is called a spider domain if it is self-similar (see (25.2)) and consists of one junction and several semi-infinite channels.

Theorem 4. *Let Ω_ε be a spider domain and $\lambda < \lambda_1$. Then*

(1) The eigenvalues $\lambda = \mu_{j,\varepsilon} = \mu_j$ of operator H_ε and the scattering matrix T do not depend on ε.

(2) The truncated resolvent (25.10) has the following estimate: if f is supported on an ε-neighborhood of the junction, then

$$|R_\lambda f| \leq C\delta^{-1}\varepsilon^{-d/2}||f||_{L^2}, \quad \lambda < \lambda_1, \quad \delta = dist(\lambda, \{\mu_j\}), \tag{25.14}$$

outside of the 2ε-neighborhood of the junction.

(3) The scattering solutions have the following form on the channels of the domain:

$$\Psi_p^{(\varepsilon)} = [\delta_{p,j}e^{-i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t} + t_{p,j}e^{i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t}]\varphi_0(y/\varepsilon) + r_{p,j}^\varepsilon, \quad x \in C_{j,\varepsilon}, \quad 1 \leq j \leq m, \tag{25.15}$$

where $|r_{p,j}^\varepsilon| \leq C\delta^{-1}e^{-\alpha\frac{t}{\varepsilon}}$ when $\varepsilon > 0$, $\frac{t}{\varepsilon} \geq 1$, and $0 \leq \lambda < \lambda_1$. Here $\alpha < \sqrt{\lambda_1 - \lambda}$ is arbitrary, $C = C(\alpha)$.

Remark 4. Formula (25.15) looks similar to the definition (25.7). In fact, the remainder in (25.7) decays only when $t \to \infty$, and (25.7) does not allow us to single out the main term of asymptotics of scattering solutions as $\varepsilon \to 0$.

Proof (of Theorem 4). All the statements above follow immediately from the self-similarity of the domain Ω_ε. Namely, we make the substitution

$$x \to \frac{x - \widehat{x}}{\varepsilon} \tag{25.16}$$

(see (25.2)) and reduce problem (25.3) in Ω_ε to the problem in Ω which corresponds to $\varepsilon = 1$. These two problems have the same eigenvalues and scattering matrices. This justifies the first statement. Let v_λ, g be functions $R_\lambda f, f$ after the change of variables (25.16). From statement (2) of Theorem 3 it follows that

$$||v_\lambda||_{L^2(K)} \leq C\delta^{-1}||g||_{L^2} = C\delta^{-1}\varepsilon^{-d/2}||f||_{L^2},$$

where K consists of the parts of the channels of Ω where $1 < t < 3$. Then the standard a priori estimates for the solutions of the equation $\Delta u - \lambda u = 0$ imply the same estimate for $|v_\lambda|$ on the cross sections $t = 2$:

$$|v_\lambda| \leq C\delta^{-1}\varepsilon^{-d/2}||f||_{L^2}, \quad t = 2.$$

The latter implies the same estimate for $|v_\lambda|$ when $t > 2$ by solving the equation $\Delta u - \lambda u = 0$ in the corresponding regions of the channels of Ω

with the boundary condition at $t = 2$. This justifies the second statement of Theorem 4. The last statement can be proved absolutely similarly. We reduce the scattering problem in Ω_ε to the scattering problem in Ω and use representation (25.13) with $\varepsilon = 1$. This implies (25.7) with $\varepsilon = 1$ and the remainder term $r_{p,j}$ such that $|r_{p,j}| \leq C\delta^{-1}e^{-\alpha t}$ for $t > 1$. It remains only to make the substitution inverse to (25.16).

In spite of its simplicity, Theorem 4 allows us to obtain two very important results: small ε asymptotics of the spectrum and the resolvent $(H_\varepsilon - \lambda)^{-1}$ of H_ε for arbitrary networks of thin wave guides Ω_ε. For this purpose, we need to rewrite (25.15) in a slightly different (less explicit) form.

We denote by $\varsigma_{p,j}$ the linear combination of exponents in the square brackets in (25.15). This is a function on the edge Γ_j of the graph. Let ς_p be the column vector with components $\varsigma_{p,j}$, $1 \leq j \leq m$. Obviously, ς_p satisfies the equation

$$(\varepsilon^2 \frac{d^2}{dt^2} + \lambda - \lambda_0)\varsigma = 0. \tag{25.17}$$

We will use the notation $\Psi_p^{(\varepsilon)}$ for both the scattering solution and the column vector whose components $\Psi_{p,j}^{(\varepsilon)}$ are restrictions of the scattering solution $\Psi_p^{(\varepsilon)}$ on the channels $C_{j,\varepsilon}$, $1 \leq j \leq m$. Then (25.15) can be rewritten in vector form as

$$\Psi_p^{(\varepsilon)} = \varsigma_p \varphi_0(y/\varepsilon) + r_p^{(\varepsilon)} = [e_p e^{-i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t} + t_p e^{i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t}]\varphi_0(y/\varepsilon) + r_p^{(\varepsilon)}, \tag{25.18}$$

where $x \in \cup_{1\leq j\leq m} C_{j,\varepsilon}$, $r_p^{(\varepsilon)}$ is the vector with components $r_{p,j}^{(\varepsilon)}$, all components $e_{p,j}$ of the vector e_p are zeros except $e_{p,p}$ which is equal to one, and t_p is the pth column of the scattering matrix T. Let us construct the $m \times m$ matrix with columns $\Psi_p^{(\varepsilon)}$ and the matrix ς with columns ς_p, $1 \leq p \leq m$. As is easy to see, $\varsigma(0) = (I + T)$, $\varsigma'(0) = i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}(-I + T)$, and therefore,

$$i\varepsilon(I + T)\varsigma'(0) - \sqrt{\lambda - \lambda_0}(I - T)\varsigma(0) = 0. \tag{25.19}$$

Of course, this equality also holds for individual columns ς_p of matrix ς.

It is essential that the GC (25.19) together with some condition at infinity is equivalent to the explicit form of ς_p given by (25.18). In fact, let ς satisfy (25.17). Then

$$\varsigma = \alpha_p e^{-i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t} + \beta_p e^{i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t}$$

with some constant vectors α_p, β_p. We will say that $\varsigma = \psi_p$ is *a solution of the scattering problem on the graph* Γ with the incident wave coming through the edge Γ_p if ψ_p satisfies equation (25.17), GC (25.19), and $\alpha_p = e_p$, i.e.,

$$\psi_p = e_p e^{-i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t} + \beta_p e^{i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t}. \tag{25.20}$$

Thus, we specify the incident wave and impose the GC defined by the scattering problem in Ω_ε, but we do not specify the scattering coefficients of the outgoing wave. The next theorem shows that the scattering problem on the graph will have the same scattering coefficients as the problem on Ω_ε.

Theorem 5. *Formulas (25.15), (25.18), and* $\Psi_p^{(\varepsilon)} = \psi_p \varphi_0(y/\varepsilon) + r_p^{(\varepsilon)}$ *are equivalent.*

Proof. It was already shown that ς_p defined in (25.18) satisfies (25.19). Conversely, if we write β_p in (25.20) as $t_p + h_p$ and substitute (25.20) into (25.19), we will have $h_p = 0$, i.e., ψ_p coincides with ς_p defined in (25.18).

25.4 One-Dimensional Problem on the Graph

The spectrum of the operator H_ε and the asymptotic behavior of the resolvent will be expressed in terms of the solutions of a problem on the limiting graph Γ which is studied in this section.

Let Ω_ε be an arbitrary (bounded or unbounded) domain as described in the Introduction, and let Γ be the corresponding limiting graph. Points of Γ will be denoted by γ with t being a parameter on each edge Γ_j of the graph. We are going to introduce a special spectral problem

$$h_\varepsilon \varsigma := -\varepsilon^2 \frac{d^2}{dt^2} \varsigma = (\lambda - \lambda_0)\varsigma \tag{25.21}$$

on smooth functions $\varsigma = \varsigma(\gamma)$ on Γ which satisfy the following GC at vertices. We split the set V of vertices v of the graph into two subsets $V = V_1 \cup V_2$, where the vertices from the set V_1 have degree 1 and correspond to the free ends of the channels, and the vertices from the set V_2 have degree at least 2 and correspond to the junctions $J_{v,\varepsilon}$. We keep the same BC at $v \in V_1$ as at the free end of the corresponding channel of Ω_ε (see (25.1)):

$$B\varsigma = 0 \quad \text{at } v \in V_1. \tag{25.22}$$

The GC at each vertex $v \in V_2$ will be defined in terms of an auxiliary scattering problem for a spider domain $\Omega'_{v,\varepsilon}$. This domain is formed by the individual junction $J_{v,\varepsilon}$ which corresponds to the vertex v, and all channels with an end at this junction, where the channels are extended to infinity if they have a finite length. Let $T = T_v(\lambda)$ be the scattering matrix for the problem (25.1) in the spider domain $\Omega'_{v,\varepsilon}$ and let I_v be the unit matrix of the same size as the size of T. We choose the parametrization on Γ in such a way that $t = 0$ at v for all edges adjacent to this particular vertex. Let $d = d(v) \geq 2$ be the order (the number of adjacent edges) of the vertex $v \in V_2$. For any function ς on Γ, we form a column vector $\varsigma^{(v)} = \varsigma^{(v)}(t)$ with $d(v)$ components which is formed by the restrictions of ς on the edges of Γ adjacent to v. We will need this vector only for small values of $t \geq 0$. The components of the vector $\varsigma^{(v)}$ are taken in the same order as the order of channels of $\Omega'_{v,\varepsilon}$. The GC at the vertex $v \in V_2$ has the form

$$i\varepsilon[I_v + T_v(\lambda)]\frac{d}{dt}\varsigma^{(v)}(t) - \sqrt{\lambda - \lambda_0}[I_v - T_v(\lambda)]\varsigma^{(v)}(t) = 0, \quad t = 0, \quad v \in V_2, \tag{25.23}$$

if $\lambda \neq \lambda_0$. Condition (25.23) can degenerate if $\lambda = \lambda_0$, and it requires some regularization in this case.

Solutions of (25.21) have the following form:

$$\varsigma = a_j e^{i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t} + b_j e^{-i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t}, \ \gamma \in \Gamma_j.$$

If $\mathrm{Im}\lambda > 0$ and $\varsigma \in L^2(\Gamma)$, then $b_j = 0$ for infinite edges (see (25.5)). Thus, if ς satisfies equation (25.21) in a neighborhood of infinity, then

$$\varsigma = a_j e^{i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t}, \quad \gamma \in \Gamma_j, \quad 1 \leq j \leq m, \quad t >> 1. \tag{25.24}$$

We will assume that condition (25.24) holds also when λ is real, i.e., we consider only those solutions of (25.21) with real $\lambda = \lambda' > \lambda_0$ which can be obtained as the limit of solutions with complex $\lambda = \lambda' + i\varepsilon$ when $\varepsilon \to 0$.

We will call function $g = g_\lambda(\gamma, \xi; \varepsilon)$, $\gamma, \xi \in \Gamma$, *the Green function of the problem (25.21)–(25.24)* if it satisfies the equation (with respect to variable γ)

$$-\varepsilon^2 \frac{d^2}{dt^2} g - (\lambda - \lambda_0)g = \delta_\xi(\gamma), \tag{25.25}$$

and conditions (25.22)–(25.24). Here ξ is a point of Γ which is not a vertex, and $\delta_\xi(\gamma)$ is the delta function supported on $\gamma = \xi$.

Lemma 1. *Let $\lambda < \lambda_1$, $\lambda \neq \lambda_0$. Operator $h_\varepsilon = -\varepsilon^2 \frac{d^2}{dt^2}$ is symmetric on the space of smooth, compactly supported functions on Γ which satisfy conditions (25.22) and (25.23).*

Proof. One needs only to show that

$$\left\langle \frac{d}{dt}\varsigma_1^{(v)}(t), \varsigma_2^{(v)}(t) \right\rangle - \left\langle \varsigma_1^{(v)}(t), \frac{d}{dt}\varsigma_2^{(v)}(t) \right\rangle = 0, \quad t = 0, \quad v \in V_2, \tag{25.26}$$

for any two vector functions $\varsigma = \varsigma_1^{(v)}, \varsigma = \varsigma_2^{(v)}$ which satisfy GC (25.23) (similar relation at $v \in V_1$ obviously holds). Let $\lambda \in (\lambda_0, \lambda_1)$. Then matrix $T_v(\lambda)$ is unitary (Theorem 2). If matrix $I_v + T_v$ is nondegenerate, we rewrite (25.23) in the form $\frac{d}{dt}\varsigma^{(v)}(t) = A\varsigma^{(v)}(t)$, $t = 0$, where the matrix

$$A = \frac{\sqrt{\lambda - \lambda_0}}{i\varepsilon}[I_v + T_v(\lambda)]^{-1}[I_v - T_v(\lambda)]$$

is real. The latter immediately implies (25.26). Similar arguments can be used if $I_v - T_v$ is nondegenerate. If both matrices are degenerate (i.e., T_v has both eigenvalues, ± 1), we consider a unitary matrix U such that UT_vU^* is a diagonal unitary matrix. Since $\langle U\varsigma_1, U\varsigma_2\rangle = \langle \varsigma_1, \varsigma_2\rangle$ for any two vectors ς_1, ς_2, one can easily reduce the proof of (25.26) to the case when T_v is a diagonal unitary matrix. Then (25.23) implies the following relations for coordinates $\varsigma_j(t)$ of the vector $\varsigma^{(v)}(t)$: $\varsigma_j'(0) = a_j\varsigma_j(0)$ or $\varsigma_j(0) = b_j\varsigma_j'(0)$, where constants a_j, b_j are real. The first case occurs if the corresponding diagonal element of T_v differs from -1, the second relation is valid if this element is -1. These relations for $\varsigma_j(t)$ imply (25.26). Similar arguments can be used to prove (25.26) when $\lambda < \lambda_0$, since matrix T_v is orthogonal in this case (see Theorem 3).

Theorem 6. *For any $\varepsilon > 0$ there is a discrete set $\Lambda(\varepsilon)$ on the interval $[-\lambda_0, \lambda_1)$ such that the Green function $g_\lambda(\gamma, \xi; \varepsilon)$ exists for all $\lambda < \lambda_1$, $\lambda \notin \Lambda(\varepsilon)$, and has the form*

$$g_\lambda = \frac{h(\gamma, \xi, \lambda, \varepsilon)}{D(\lambda, \varepsilon)}, \tag{25.27}$$

where function h is continuous on the set $\gamma, \xi \in \Gamma$, $\lambda < \lambda_1$, $\varepsilon > 0$ and uniformly bounded on each bounded subset, and

$$D(\lambda, \varepsilon) = \Sigma_{m=1}^{N} c_m(\lambda) e^{i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}s_m}. \tag{25.28}$$

Here s_m are constants, functions $c_m(\lambda)$ are analytic in $\lambda < \lambda_1$ with a branch point of second order at $\lambda = \lambda_0$, and $D \neq 0$ if $\lambda < \lambda_0$.

Proof. We fix the parametrization on each edge Γ_j of the graph. Then, obviously,

$$g_\lambda = a_j e^{-i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t} + b_j e^{i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t}, \quad \gamma \in \Gamma_j, \quad \text{if } \xi \notin \Gamma_j, \tag{25.29}$$

$$g_\lambda = a_j e^{-i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t} + b_j e^{i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t} + \frac{\varepsilon}{\sqrt{\lambda-\lambda_0}}\sin[\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}(t-\tau)_-], \quad \text{if } \xi \in \Gamma_j. \tag{25.30}$$

Here τ is the coordinate of the point ξ, $(t-\tau)_- = \min(t-\tau, 0)$, and the last term in (25.30) is a particular solution of (25.25) on Γ_j with a bounded support. There are $2N$ unknown constants in the formulas above, where N is the total number of edges of the graph. Conditions (25.22)–(25.24) provide $2N$ linear equations for these constants. As is easy to see, the coefficients for unknowns in all the equations have the form $a(\lambda)e^{i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}s}$, where $a(\lambda)$ is analytic in $\lambda < \lambda_1$ with a branch point of second order at $\lambda = \lambda_0$, and $s = 0$ or $\pm l_j$ (l_j are the lengths of the finite channels). The exponential factors in the coefficients appear when the formulas (25.29), (25.30) are substituted into GC at the end point of the edge Γ_j where $t = l_j$. We apply Cramer's rule to solve this system of $2N$ equations. This immediately provides all the statements of the theorem with $D(\lambda, \varepsilon)$ being the determinant of the system. One only needs to show that $D \neq 0$ for $\lambda < \lambda_0$. Note that the latter fact implies the discreteness of the set $\Lambda(\varepsilon) = \{\lambda : D(\lambda, \varepsilon) = 0\}$.

Obviously, $D = 0$ if and only if the homogeneous problem (25.21)–(25.24) has a non-trivial solution. Let $\lambda < \lambda_0$. Then solutions ς of the problem (25.21)–(25.24) decay at infinity, and

$$\begin{aligned} 0 &= \int_\Gamma [-\varepsilon^2 \varsigma'' - (\lambda - \lambda_0)\varsigma]\varsigma d\gamma \\ &= -\varepsilon^2 \Sigma_v \left\langle \frac{d}{dt}\varsigma^{(v)}, \varsigma^{(v)} \right\rangle |_v + \int_\Gamma [\varepsilon^2 (\varsigma')^2 - (\lambda - \lambda_0)\varsigma^2] d\gamma. \quad (25.31) \end{aligned}$$

It was shown in the proof of Lemma 1 that it is enough to consider only diagonal matrices T when the terms under the sign Σ_v above with $v \in V_2$ are evaluated. Since T is orthogonal when $\lambda < \lambda_0$, the diagonal elements of T are equal to ± 1. Then (25.23) means that each component of the vector $\varsigma^{(v)}$ or its derivative is zero at the vertex. Hence, the terms in the sum above with $v \in V_2$ are equal to zero. They are zeros also for those $v \in V_1$ where the boundary condition in (25.22) is the Dirichlet or Neumann condition. If $v \in V_1$ and $B = \varepsilon \frac{d}{dt} + a, \ a \geq 0$, these terms are nonpositive. Hence, relation (25.31) implies that $\varsigma = 0$ when $\lambda < \lambda_0$.

Theorem 6 does not contain a statement concerning the structure of the discrete set $\Lambda(\varepsilon)$. This set becomes more and more dense when $\varepsilon \to 0$. In general, every point $\lambda' \in (\lambda_0, \lambda_1)$ belongs to $\Lambda(\varepsilon)$ for some sequence of $\varepsilon = \varepsilon_j(\lambda') \to 0$. However, it is not an absolutely arbitrary discrete set, but the set of zeros of a specific analytic function (25.28), and this fact provides the following restriction on the set $\Lambda(\varepsilon)$.

Lemma 2. *For each bounded interval* $[\alpha, \lambda_1]$, *each* $\sigma > 0$ *and some* M, *there are* $c\varepsilon^{-1}$ *intervals* I_j *of length* σ *such that*

$$|D(\lambda, \varepsilon)| > c\sigma^M \quad \textit{when} \ \ \varepsilon > 0, \quad \lambda \in [\alpha, \lambda_1] \backslash \cup I_j, \quad c = c(\alpha).$$

This lemma is a particular case of Lemma 15 from [MoVa07] (the set Γ_0 is empty in the case considered here).

In order to construct the resolvent of the problem in Ω_ε, we need to represent the Green function g_λ of the problem (25.25), (25.22)–(25.24) on the graph Γ through the solutions of the scattering problems on the spider subgraphs of Γ.

We will call a function $\psi = \psi_p(\gamma)$ a *solution of the scattering problem on the graph* Γ if it satisfies the equation (25.21), conditions (25.22), and (25.23), and has the following form at unbounded edges of the graph:

$$\psi_p(\gamma) = \delta_{p,j} e^{-i\frac{\sqrt{\lambda - \lambda_0}}{\varepsilon}t} + a_{p,j} e^{i\frac{\sqrt{\lambda - \lambda_0}}{\varepsilon}t}, \ \ \gamma \in \Gamma_j, \quad 1 \leq p, j \leq m, \quad (25.32)$$

where $\delta_{p,j}$ is the Kronecker symbol. This scattering solution corresponds to the wave coming through the edge Γ_p. These scattering solutions on the graphs were introduced in the previous section in the case when the graph corresponds to a spider domain. In fact, only this simple case will be needed below.

Lemma 3. *If the graph Γ corresponds to a spider domain Ω_ε, then the scattering solution $\psi_p(\gamma)$ exists and is defined uniquely for all $\lambda < \lambda_1$, $\lambda \neq \lambda_0$. Any function ς on Γ which satisfies equation (25.21) and GC condition (25.23) is a linear combination of the scattering solutions $\psi_p(\gamma)$.*

Remark 5. For arbitrary graphs, one may have non-trivial solutions of the homogeneous problem (25.21)–(25.24) supported on the set of bounded edges of the graph. This occurs when $\lambda \in \Lambda(\varepsilon)$. The set $\Lambda(\varepsilon)$ is empty for spider graphs.

Proof (of Lemma 3). If we take $a_{p,j} = t_{p,j}$, where $t_{p,j}$ are the scattering coefficients in the spider domain Ω_ε, then function (25.32) will satisfy (25.23) (see the derivation of (25.19)). Hence, the scattering solutions $\psi_p(\gamma)$ exist for all $\lambda < \lambda_1$, $\lambda \neq \lambda_0$, since the scattering coefficients are defined for those λ by Theorem 3. If we put function (25.32) with $a_{p,j} = t_{p,j} + h_{p,j}$ into GC (25.23), we immediately get that $h_{p,j} = 0$ (see the proof of Theorem 5). Thus, scattering solutions are defined uniquely. The space of solutions of equation (25.21) is $2m$ dimensional. The $(m \times 2m)$-dimensional matrix $(I_v + T_v(\lambda), I_v + T_v(\lambda))$ formed from coefficients in GC (25.23) has rank m. Hence, the solution space of the problem (25.21), (25.23) is m dimensional. Obviously, functions ψ_p are linearly independent on Γ. Thus, any solution of (25.21), (25.23) is a linear combination of functions ψ_p.

Let Γ_{j_0} be the edge of Γ which contains the point ξ (see (25.25)). We cut the graph Γ into simple graphs $\Gamma(v)$ with one vertex v by cutting all the bounded edges at some points $\xi_j \in \Gamma_j$. We will choose $\xi_{j_0} = \xi$. Let us denote by $\Gamma'(v)$ the spider graph which is obtained by extending all the edges of $\Gamma(v)$ to infinity. Let $\psi_{p,v}(\gamma)$ be the scattering solutions on the graph $\Gamma'(v)$.

Lemma 4. *There exist functions*

$$a = a_{p,v}(\lambda, \varepsilon, \xi), \quad \lambda < \lambda_1, \ \varepsilon > 0, \ \xi \in \Gamma_{j_0},$$

which are continuous, bounded on each bounded set, and such that

$$g_\lambda = \Sigma_p \frac{a_{p,v}(\lambda, \varepsilon, \xi)}{D(\lambda, \varepsilon)} \psi_{p,v}(\gamma), \quad \gamma \in \Gamma(v).$$

Proof. It follows from the previous lemma that g_λ can be represented as a linear combination of the scattering solutions:

$$g_\lambda = \Sigma_p c_{p,v} \psi_{p,v}(\gamma), \quad \gamma \in \Gamma(v).$$

In order to find the coefficients $c_{p,v}$, we note that g_λ is equal to a combination of two exponents on the edge $\Gamma_p \subset \Gamma(v)$ with the coefficient of the incident wave equal to $c_{p,v}$:

$$g_\lambda = c_{p,v} e^{i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t} + b_{p,v} e^{-i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t}, \quad \gamma \in \Gamma_p \subset \Gamma(v).$$

Now $c_{p,v}$ can be found by comparing the formula above and (25.27) at two points of Γ_p.

25.5 Small ε Asymptotics for the Problem in Ω_ε

As everywhere above, the domain Ω_ε, considered below, can be bounded or unbounded. Denote by Λ^0 the union of eigenvalues of the operator (25.3) in all the spider domains $\Omega'_{v,\varepsilon}$ associated to Ω_ε. These spider domains consist of individual junctions and all the channels adjacent to this junction. The channels are extended to infinity if they have a finite length. The set Λ^0 does not depend on ε due to Theorem 4. Let us recall that $\Lambda(\varepsilon)$ is the set of eigenvalues of the one-dimensional problem (25.21)–(25.24) on the limiting graph (see Theorem 6).

The eigenvalues of the operator $H_\varepsilon = -\varepsilon^2\Delta$ of the problem (25.1) which are located on the interval $(-\infty, \lambda_1)$ are exponentially close to the set $\Lambda^0 \cup \Lambda(\varepsilon)$. In the process of proving this statement, we will get the asymptotic approximation of the resolvent $(H_\varepsilon - \lambda)^{-1}$ as $\varepsilon \to 0$. Namely, the following theorem will be proved.

Let $\lambda' < \lambda_1$ and let Λ^ν be an $e^{\frac{-\nu\alpha}{\varepsilon}}$-neighborhood of the set $\Lambda^0 \cup \Lambda(\varepsilon)$. Assume that the right-hand side of (25.3) has a compact support which is separated from junctions, i.e., there exist $\tau, d > 0$ such that the support of f belongs to $\cup\Delta_j$, where Δ_j is the part of the channel $C_{j,\varepsilon}$ defined by the inequalities $\tau \le t \le l_j - \tau$ if $l_j < \infty$, or $\tau \le t \le d$ if the channel is infinite.

Theorem 7. *(1) There exists $\nu > 0$ such that the eigenvalues $\mu_{j,\varepsilon}$ of the operator H_ε which belong to the interval $(-\infty, \lambda')$ with an arbitrary $\lambda' < \lambda_1$ are located in an $e^{\frac{-\nu\alpha}{\varepsilon}}$-neighborhood of the set $\Lambda^0 \cup \Lambda(\varepsilon)$. Here $\alpha = \lambda_1 - \lambda'$.*

(2) Let the support of f belong to $\cup\Delta_j$ and let $u = R_\lambda f$ be the solution of problem (25.3). Here R_λ is the truncated resolvent (25.9). Then for any $\eta > 0$, there exist $\nu > 0$ and $\rho = \rho(\eta) > 0$ such that $u = R_\lambda f$ has the following asymptotic behavior in all the channels outside the η-neighborhood of the support of f:

$$u = R_\lambda f = (\widehat{g}_\lambda f_0)\varphi_0(\frac{y}{\varepsilon}) + O(e^{\frac{-\rho}{\varepsilon}}), \qquad \lambda \in (-\infty, \lambda')\backslash\Lambda^\nu, \quad \varepsilon \to 0. \quad (25.33)$$

Here

$$f_0 = f_0(\gamma) = \left\langle f, \varepsilon^{-d/2}\varphi_0(\frac{y}{\varepsilon})\right\rangle, \quad \gamma \in \Gamma,$$

and $\widehat{g}_\lambda$ *is the integral operator on the graph* Γ *whose kernel is the Green function* g_λ *constructed in Theorem 6:*

$$\widehat{g}_\lambda f_0 = \int_\Gamma g_\lambda(\gamma,\xi;\varepsilon)f_0(\xi)d\xi.$$

Remark 6. Below, we also will get the asymptotics of $u = R_\lambda f$ on the support of f, as well as a more precise estimate of the remainder in (25.33).

Proof (of Theorem 7). Let

$$f_1 = f_1(x) = f - \varepsilon^{-d/2} f_0 \varphi_0(\frac{y}{\varepsilon}), \quad x \in \Omega_\varepsilon,$$

i.e., $f_0 = f_0(\gamma)$ is the first Fourier coefficient of the expansion of f with respect to the basis $\{\varepsilon^{-d/2}\varphi_j(\frac{y}{\varepsilon})\}$, and f_1 is the sum of all the terms of the expansion without the first one. We are going to show that $u = R_\lambda f$ has the following form on the channels of Ω_ε:

$$u = R_\lambda f = (\widehat{g}_\lambda f_0)\varphi_0(\frac{y}{\varepsilon}) + \chi R^0_\lambda f_1 + O(e^{\frac{-\rho}{\varepsilon}}), \quad \lambda \in (-\infty,\lambda')\backslash \Lambda^\nu, \ \ \varepsilon \to 0, \tag{25.34}$$

where $\nu, \rho > 0$, $\chi \in C^\infty(\Omega_\varepsilon)$ is a cut-off function such that $\chi = 0$ on all the junctions, $\chi = 1$ outside of the ε-neighborhood of junctions, and function $R^0_\lambda f_1$ is defined by solving the following simple problem in the infinite cylinder. Let $f_{1,j}$ be the restriction of f_1 onto the channel $C_{j,\varepsilon}$. We extend the channel $C_{j,\varepsilon}$ to infinity (in both directions) and extend $f_{1,j}$ by zero. Let u_j be the outgoing solution of the equation

$$-\varepsilon^2 \Delta u - \lambda u = f_{1,j}$$

in the extended channel. Then $R^0_\lambda f_1$ is defined as $R^0_\lambda f_1 = u_j$ in the channel $C_{j,\varepsilon}$. Obviously, $\chi R^0_\lambda f_1$ can be considered as a function on Ω_ε.

The justification of (25.34) and the proof of Theorem 7 are based on an appropriate choice of the parametrix ("almost inverse operator"):

$$P_\lambda : L^2_{\tau,d} \to L^2_{loc}(\Omega_\varepsilon),$$

which is defined as follows:

$$P_\lambda f = (\widehat{G}_\lambda f_0)\varphi_0(\frac{y}{\varepsilon}) + (\chi R^0_\lambda f_1) - \Sigma_v \chi_v R^0_{\lambda,v}[\chi_v[(\varepsilon^2\Delta + \lambda)(\chi R^0_\lambda f_1) - f_1]]. \tag{25.35}$$

Here $L^2_{\tau,d}$ is a subspace of $L^2(\Omega_\varepsilon)$ which consists of functions supported on $\cup \Delta_j$. Now we are going to define and study, successively, each of the terms in the formula above. In particular, we need to show that

$$-(\varepsilon^2\Delta + \lambda)P_\lambda f = f + Q_\lambda f, \quad Q_\lambda : L^2_{\tau,d} \to L^2_{\tau,d}, \quad ||Q_\lambda|| \le Ce^{\frac{-\rho}{\varepsilon}}. \tag{25.36}$$

Operator $\widehat{G}_\lambda$ is an integral operator with the kernel $G_\lambda(x,z;\varepsilon)$, $x, z \in \Omega_\varepsilon$, which is defined as follows. We split Ω_ε into domains $\Omega_{v,\varepsilon}$ by cutting all the finite channels $C_{j,\varepsilon}$ using the cross sections $t = t_j$. Let $z \in \Delta_{j_0}$. Then we choose t_{j_0} to be equal to the coordinate $t = t(z)$ of the point z. Other cross sections are chosen with the only condition that $\tau < t_j < l_j - \tau$, i.e., the cross section $t = t_j$ is strictly inside of Δ_j. Let $\Omega'_{v,\varepsilon}$ be the spider domain which we get by extending all the finite channels of $\Omega_{v,\varepsilon}$ to infinity. Let $\Psi^{(\varepsilon)}_{p,v}$ be the scattering solutions of the problem in the spider domain $\Omega'_{v,\varepsilon}$. The small ε asymptotics of these solutions is given by Theorems 4 and 5. We introduce the following functions $\widetilde{\Psi}^{(\varepsilon)}_{p,v}$ by modifying the remainder terms in these asymptotics:

$$\widetilde{\Psi}^{(\varepsilon)}_{p,v} = \psi_p \varphi_0(y/\varepsilon) + \chi_v r^{(\varepsilon)}_p, \tag{25.37}$$

where $\chi_v \in C^\infty(\Omega_\varepsilon)$, $\chi_v = 1$ on a τ-neighborhood of the junction, and $\chi_v = 0$ outside of $\Omega_{v,\varepsilon}$. Then we define G_λ by the formula

$$G_\lambda(x, z; \varepsilon) = \Sigma_p \frac{a_{p,v}(\lambda, \varepsilon, \xi)}{D(\lambda, \varepsilon)} \widetilde{\Psi}^{(\varepsilon)}_{p,v}, \qquad x \in \Omega_{v,\varepsilon}, \tag{25.38}$$

where $a_{p,v}, D$ are defined in Lemma 4, and ξ is the point on the graph Γ which corresponds to $z \in \Delta_{j_0}$, i.e., the point on the edge Γ_{j_0} where $t = t_{j_0}$. Since function $\Psi^{(\varepsilon)}_{p,v}$ satisfies the equation $(\varepsilon^2\Delta + \lambda)u = 0$ on $\Omega_{v,\varepsilon}$, from Theorems 4 and 5 it follows that

$$-(\varepsilon^2\Delta + \lambda)\widetilde{\Psi}^{(\varepsilon)}_{p,v} = O(\delta^{-1} e^{-\frac{\alpha\tau}{\varepsilon}}), \quad \varepsilon \to 0, \quad -\infty < \lambda < \lambda' \ , \ x \in \Omega_{v,\varepsilon},$$

where $\alpha = \lambda_1 - \lambda'$. We choose $\nu < \frac{\tau}{4}$. Then $\delta > e^{-\frac{\alpha\tau}{4\varepsilon}}$ for $\lambda \in (-\infty, \lambda') \setminus \Lambda^\nu$, and

$$-(\varepsilon^2\Delta + \lambda)\widetilde{\Psi}^{(\varepsilon)}_{p,v} = O(e^{-\frac{3\alpha\tau}{4\varepsilon}}), \quad \varepsilon \to 0, \quad \lambda \in (-\infty, \lambda') \setminus \Lambda^\nu, \ x \in \Omega_{v,\varepsilon}.$$

Since coefficients $a_{p,v}$ are bounded, Lemma 2 with $\sigma = e^{-\frac{\alpha\tau}{4M\varepsilon}}$ implies that

$$-(\varepsilon^2\Delta + \lambda)G_\lambda = O(e^{-\frac{\alpha\tau}{2\varepsilon}}), \quad \varepsilon \to 0, \ \lambda \in (-\infty, \lambda') \setminus \Lambda^\nu, \ x \in \Omega_{v,\varepsilon}. \tag{25.39}$$

Relations (25.39) are valid on each domain $\Omega_{v,\varepsilon}$. Now we are going to combine them and evaluate $(\varepsilon^2\Delta+\lambda)G_\lambda$ for all $x \in \Omega_\varepsilon$. From (25.37), (25.38), and Lemma 4, it follows that the function

$$G_\lambda - g_\lambda(\gamma, \xi; \varepsilon)\varphi_0(\frac{y}{\varepsilon})$$

is infinitely smooth in the channels of Ω_ε. Here γ is the point on Γ which corresponds to $x \in \Omega_\varepsilon$. Then from (25.39) it follows that

$$-(\varepsilon^2\Delta+\lambda)G_\lambda = \delta_\xi(\gamma)\varphi_0(\frac{y}{\varepsilon}) + O(e^{-\frac{\alpha\tau}{2\varepsilon}}), \quad \varepsilon \to 0, \ \lambda \in (-\infty, \lambda') \setminus \Lambda^\nu, \ x \in \Omega_\varepsilon. \tag{25.40}$$

As is easy to see, the remainder in (25.40) is zero in the region where $\nabla\chi_v \neq 0$, i.e., the support of the remainder belongs to $\cup\Delta_j$.

Now let us study the second and third terms on the left-hand side of (25.35). Obviously,

$$-(\varepsilon^2\Delta+\lambda)(\chi R^0_\lambda f_1)=\chi f_1+h=f_1+h,\quad h=-2\varepsilon^2\nabla\chi\cdot\nabla R^0_\lambda f_1-\varepsilon^2(\Delta\chi)R^0_\lambda f_1. \tag{25.41}$$

Here we used the fact that $\chi=1$ on the support of f_1. Since f_1 is orthogonal to $\varphi_0(\frac{y}{\varepsilon})$, function $R^0_\lambda f_1$ and all its derivatives decay exponentially in each channel $C_{j,\varepsilon}$ as $\frac{r}{\varepsilon}\to\infty$, where r is the distance from Δ_j. Hence,

$$h=O(e^{-\frac{\alpha(\tau-\varepsilon)}{\varepsilon}})=O(e^{-\frac{\alpha\tau}{\varepsilon}}),\quad \varepsilon\to 0,\ \ \lambda\in(-\infty,\lambda'). \tag{25.42}$$

The remainder terms will be parts of the operator Q_λ, and we need the kernel of this operator to be supported on $\cup\Delta_j$. Unfortunately, h is supported on ε-neighborhoods of the junctions. The last term in (25.35) is designed to correct this. Since h is supported on the region where $\nabla\chi\neq 0$, function h can be represented as the sum $h=\Sigma_v h_v$, where $h_v=\chi_v h$ has estimate (25.42) and is supported on the ε-neighborhood of the junction $J_{v,\varepsilon}$ which corresponds to the vertex v. Consider $\widetilde{h}=\Sigma_v\chi_v R^0_{\lambda,v}[\chi_v h]$, which is defined as follows. We apply the resolvent $R^0_{\lambda,v}$ of the problem in the spider domain $\Omega'_{v,\varepsilon}$ to h_v, multiply the result by χ_v, and extend the product by zero on $\Omega_\varepsilon\backslash\Omega_{v,\varepsilon}$.

From (25.42) and Theorem 4 it follows that

$$|R^0_{\lambda,v}h_v|\le C\delta^{-1}e^{-\frac{\alpha\tau}{\varepsilon}}\le Ce^{-\frac{\alpha\tau}{2\varepsilon}},\quad \varepsilon\to 0,\ \lambda\in(-\infty,\lambda')\backslash\ \Lambda^\nu, \tag{25.43}$$

if we choose $\nu<\frac{\tau}{2}$, so that $\delta>e^{-\frac{\alpha\tau}{2\varepsilon}}$. From standard a priori estimates for the solutions of homogeneous equation $(\varepsilon^2\Delta+\lambda)u=0$, it follows that estimate (25.14) is valid also for all derivatives of $R_\lambda f$, since this function satisfies the homogeneous equation outside of the 2ε-neighborhood of the junction. Then (25.43) holds for the derivatives of $R^0_{\lambda,v}h_v$. This allows us to obtain, similarly to (25.41), that

$$-(\varepsilon^2\Delta+\lambda)\widetilde{h}=h+h_1,\quad h_1=O(e^{-\frac{\alpha\tau}{2\varepsilon}}),\quad \varepsilon\to 0,\ \ \lambda\in(-\infty,\lambda')\backslash\ \Lambda^\nu, \tag{25.44}$$

where h_1 is supported on the closure of the set $\nabla\chi_v\neq 0$. This set belongs to $\cup\Delta_j$. Finally, from (25.40), (25.41), (25.44) it follows that

$$-(\varepsilon^2\Delta+\lambda)P_\lambda f=f+g,\quad g=O(e^{-\frac{\varrho}{\varepsilon}}),\quad \varepsilon\to 0,\ \ \lambda\in(-\infty,\lambda')\backslash\ \Lambda^\nu, \tag{25.45}$$

and g is supported on $\cup\Delta_j$. One can easily check that g depends linearly on f. Besides, one can specify the dependence on the norm of f in estimates of all the remainders above. This will lead to (25.36) instead of (25.45). In fact, (25.36) is valid when Q_λ is considered as an operator in L^2 or as an operator in the space of continuous functions on $\cup\Delta_j$.

We are now going to construct the solution u of problem (25.3) with $f\in L^2_{\tau,d}$. We look for u in the form $u=P_\lambda g$ with unknown $g\in L^2_{\tau,d}$. Obviously, u satisfies the boundary conditions and appropriate conditions at infinity.

Equation (25.3) in Ω_ε leads to $g + Q_\lambda g = f$. Since the norm of operator Q_λ is exponentially small, function g exists, is unique, and $g = f + q$, $||q|| \le Ce^{-\frac{\rho}{\varepsilon}}||f||$, i.e.,

$$u = P_\lambda(f+q), \quad ||q||_{L^2_{\tau,d}} \le Ce^{-\frac{\rho}{\varepsilon}}||f||_{L^2_{\tau,d}}, \quad \varepsilon \to 0, \;\; \lambda \in (-\infty, \lambda')\setminus \Lambda^\nu.$$

This justifies (25.34) and (25.33). The first statement of Theorem 7 follows from here. Namely, assume that an eigenvalue $\mu = \mu_{j,\varepsilon}$ of the operator H_ε belongs to $(-\infty, \lambda')\setminus \Lambda^\nu$. Then the truncated resolvent R_λ (see (25.9)) has a pole there (see Theorem 3). The residue of this pole is the orthogonal projection on the eigenspace of H_ε. The pole of $R_\lambda f$ may disappear only if f is orthogonal to the eigenspace which corresponds to the eigenvalue $\lambda = \mu$. Non-trivial solutions of the equation $(\Delta + \lambda)u = 0$ in Ω_ε cannot be equal to zero in a subdomain of Ω_ε. Thus, there is a function $f \in L^2_{\tau,d}$ which is not orthogonal to the eigenspace, and $R_\lambda f$ must have a pole at $\lambda = \mu$. This contradicts (25.34) and (25.33).

The following statement can be easily proved using Theorem 7 and reduction (25.13) of the scattering problem to problems (25.3), (25.4).

Theorem 8. *For any interval $[\alpha, \lambda')$, there exist $\rho, \nu > 0$ such that scattering solutions $\Psi_{p,\varepsilon}(x)$ of the problem in $\Omega\varepsilon$ have the following asymptotic behavior on the channels of Ω_ε as $\varepsilon \to 0$:*

$$\Psi(x) = \psi_{p,\varepsilon}(\gamma)\varphi_0(\frac{y}{\varepsilon}) + r_p^{(\varepsilon)}(x),$$

where $\psi_p(\gamma) = \psi_p^{(\varepsilon)}(\gamma)$ are the scattering solutions of the problem on the graph Γ and

$$|r_p^{(\varepsilon)}(x)| \le Ce^{\frac{-\rho d(\gamma)}{\varepsilon}}, \qquad \lambda \in [\alpha, \lambda')\setminus \Lambda^\nu.$$

Here $\gamma = \gamma(x)$ is the point on Γ which is defined by the cross section of the channel through the point x, and $d(\gamma)$ is the distance between γ and the closest vertex of the graph.

25.6 Eigenvalues Near the Threshold

In some cases, in particular when the parabolic problem is studied, the lower part of the spectrum of the operator H_ε is of particular importance. Theorem 7 provides a full description of the location of the eigenvalues. They are situated in an exponentially small neighborhood of $\Lambda^0 \cup \Lambda(\varepsilon)$. The set Λ^0 is determined by the junctions. The points from Λ^0 are ε-independent, non-negative and may be located on either or both sides of λ_0. One may have at most a finite number of eigenvalues below λ_0. The points of $\Lambda(\varepsilon)$ are eigenvalues of the one-dimensional problem (25.21)–(25.24) on the limiting graph, they cannot occur below λ_0, since $D \neq 0$ there (see Theorem 6).

We are going to study the limiting behavior, as $\varepsilon \to 0$, of points from the set $\Lambda(\varepsilon)$ located in an $O(\varepsilon^2)$ neighborhood of λ_0.

We will assume that Ω_ε has at least one bounded channel (for example, Ω_ε is bounded). The opposite case is studied in Theorem 4. We also assume that $\lambda = \lambda_0 + O(\varepsilon^2)$. Then the eigenvalues of the problem (25.21)–(25.24) will depend on the form of the GC (25.23) at $\lambda = \lambda_0$. Let us put $\lambda = \lambda_0 + \mu\varepsilon^2$ in (25.21)–(25.24). Then this problem takes the form

$$-\frac{d^2}{dt^2}\varsigma = \mu\varsigma \qquad \text{on } \Gamma, \tag{25.46}$$

$$B\varsigma = 0 \quad \text{at } v \in V_1, \tag{25.47}$$

$$i[I_v+T_v(\lambda_0+\mu\varepsilon^2)]\frac{d}{dt}\varsigma^{(v)}(t)-\mu[I_v-T_v(\lambda_0+\mu\varepsilon^2)]\varsigma^{(v)}(t) = 0, \qquad t=0, \qquad v \in V_2, \tag{25.48}$$

$$\varsigma = a_j e^{i\mu t}, \quad \gamma \in \Gamma_j, \quad 1 \le j \le m, \quad t >> 1. \tag{25.49}$$

The last condition is not needed if Ω_ε is bounded ($m = 0$).

Since matrix $T_v(\lambda_0)$ is orthogonal and its eigenvalues are ± 1, the GC (25.48) with $\varepsilon = 0$ has the form

$$P\varsigma^{(v)}(0) = 0, \quad P^\perp \frac{d}{dt}\varsigma^{(v)}(0) = 0, \qquad v \in V_2, \tag{25.50}$$

where $P, P^\perp$ are projections onto eigenspaces of matrix $T_v(\lambda_0)$ with the eigenvalues ∓ 1, respectively. Let k be the dimension of the operator P, and $d - k$ be the dimension of the operator $P^\perp$, where $d = d(v)$ is the size of the vector $\varsigma^{(v)}$. Then (25.50) imposes k Dirichlet conditions and $d - k$ Neumann conditions on the components of vector $\varsigma^{(v)}$ written in the eigenbasis of the matrix $T_v(\lambda_0)$. Note that the standard Kirchhoff conditions (ς is continues on Γ, a linear combination of derivatives is zero at each vertex) has the same nature, and $k = d - 1$ in this case.

Problem (25.46)–(25.49) with $\varepsilon = 0$ has a discrete spectrum $\{\mu_j\}$, $j \ge 1$, and the same problem with $\varepsilon > 0$ is its analytic perturbation. Thus, the following statement is valid.

Theorem 9. *If eigenvalues $\{\mu_j\}$ are simple, then eigenvalues $\{\mu_j(\varepsilon)\}$ of problem (25.46)–(25.49) are analytic in ε:*

$$\mu_j(\varepsilon) = \sum_{n\ge 0} \mu_{j,n}\varepsilon^n, \quad \mu_{j,0} = \mu_j. \tag{25.51}$$

Remark 7. 1. This statement implies that eigenvalues $\lambda \in \Lambda(\varepsilon)$ in an $O(\varepsilon^2)$-neighborhood of λ_0 have the form

$$\lambda = \lambda_j(\varepsilon) = \lambda_0 + \varepsilon^2 \sum_{n\ge 0} \mu_{j,n}\varepsilon^n. \tag{25.52}$$

2. The assumption on simplicity of μ_j often can be omitted. For example, (25.51), (25.52) remain valid without this assumption if $k = d$ (the limiting problem is the Dirichlet problem). In the latter case one may have multiple eigenvalues (for example, when the graph has edges of multiple lengths), but the problem with $\varepsilon = 0$ is split into separate problems on individual edges.

Theorem 9 makes it important to specify the value of k in the condition (25.50). This value depends essentially on the type of the boundary conditions at $\partial\Omega_\varepsilon$ and on whether $\lambda = \lambda_0$ is a pole of the truncated resolvent (25.10) or not.

Definition 3. *A ground state of the operator H_ε in a domain Ω_ε at $\lambda = \lambda_0$ is the function $\psi_0 = \psi_0(x)$, which is bounded, strictly positive inside Ω_ε, satisfies the equation $(-\Delta - \lambda_0)\psi_0 = 0$ in Ω_ε, and the boundary condition on $\partial\Omega_\varepsilon$, and has the following asymptotic behavior at infinity:*

$$\psi_0(x) = \varphi_0\left(\frac{y}{\varepsilon}\right)[\rho_j + o(1)], \quad x \in C_j, \quad |x| \to +\infty, \tag{25.53}$$

where $\rho_j > 0$ and φ_0 is the ground state of the operator in the cross sections of the channels.

Obviously, if the Neumann boundary condition is imposed on $\partial\Omega_\varepsilon$, then $\lambda_0 = 0$, and the ground state at $\lambda = 0$ exists and equal to a constant. It was shown in [MoVa07], [MoVa08] that the ground state at $\lambda = \lambda_0$ does not exist for generic domains Ω_ε in the case of other boundary conditions on $\partial\Omega_\varepsilon$. In particular, it does not exist if there are eigenvalues of H_ε below λ_0, or if the truncated resolvent does not have a pole at $\lambda = \lambda_0$. The following result was proved in [MoVa07] and [MoVa08].

Theorem 10. *(1) The ground state at $\lambda = \lambda_0$ implies $k = d - 1$. Thus, the eigenvalues $\mu_j(\varepsilon)$, $\varepsilon \to 0$, converge to the eigenvalues of the Kirchhoff problem in the case of the Neumann condition on $\partial\Omega_\varepsilon$ (Ω_ε is arbitrary) and in the case of other boundary conditions on $\partial\Omega_\varepsilon$ for special, nongeneric Ω_ε.*

(2) If the Dirichlet or Robin condition is imposed on $\partial\Omega_\varepsilon$ and the truncated resolvent does not have a pole at $\lambda = \lambda_0$ (this is a generic condition on Ω_ε), then $k = d$ and $\mu_j(\varepsilon)$, $\varepsilon \to 0$, converge to the eigenvalues of the Dirichlet problem.

Other possible (nongeneric) GC at $\lambda = \lambda_0$ are given by (25.50).

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26

Homogenization of a Convection–Diffusion Equation in a Thin Rod Structure

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26.1 Introduction

This chapter is devoted to the homogenization of a stationary convection–diffusion model problem in a thin rod structure. More precisely, we study the asymptotic behavior of solutions to a boundary value problem for a convection–diffusion equation defined in a thin cylinder that is the union of two nonintersecting cylinders with a junction at the origin. We suppose that in each of these cylinders the coefficients are rapidly oscillating functions that are periodic in the axial direction, and that the microstructure period is of the same order as the cylinder diameter. On the lateral boundary of the cylinder we assume the Neumann boundary condition, while at the cylinder bases the Dirichlet boundary conditions are posed.

Similar problems for the elasticity system have been intensively studied in the existing literature. We quote here the works [KoPa92], [MuSi99], [Naz82], [Naz99], [TuAg86], [TrVi87], [Ve95]. The contact problem of two heterogeneous bars was considered in [Pa94-I], [Pa96-II], [Past02]. Elliptic equations in divergence form have been addressed, for example, in [BaPa89] and [Pa05]. In contrast to the divergence-form operators, in the case of the convection-diffusion equation the asymptotic behavior of solutions depends crucially on the direction of what is called the effective convection, which is introduced in Section 26.2. In this chapter we only consider the case when in each of the two cylinders (being the constituents of the rod) the effective convection is directed from the end of the cylinder towards the junction.

The asymptotic expansion of a solution includes the interior expansion, the boundary layers in the neighborhoods of the cylinder ends, and the interior boundary layer in the vicinity of the junction. Note that the leading term of the asymptotics is described in terms of a pair of first order ordinary differential equations. The construction of the interior expansions follows the classical scheme. The analysis of boundary layers in the neighborhoods of the cylinder ends relies on the results obtained in [PaPi09]. In order to build the interior boundary layer we study a qualitative problem for the convection–diffusion

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equation in an infinite cylinder. This is done in Section 26.7. As far as the authors are aware, no one has studied a convection–diffusion equation with first order terms in an infinite cylinder. In the case under consideration, when in each of the two cylinders the effective convection is directed from the end of the cylinder towards the junction, we prove the existence of a solution for such a problem and discuss its qualitative properties. In other cases the situation is much more difficult (especially in the case when effective convections occur in opposite directions) and outside the scope of the present work.

26.2 Problem Statement

Let Q be a bounded $C^{2,\alpha}$ domain in $(d-1)$-dimensional Euclidean space $\mathbb{R}^{d-1}$ with points $x' = (x_2, ..., x_d)$. Denote $G_\varepsilon = [-1,1] \times (\varepsilon Q) \subset \mathbb{R}^d$ a thin rod with the lateral boundary $\varGamma_\varepsilon = [-1,1] \times \partial(\varepsilon Q); x = (x_1, x')$. We study the homogenization of a scalar elliptic equation with periodically oscillating coefficients

$$\begin{cases} A^\varepsilon u^\varepsilon \equiv -\mathrm{div}\left(a^\varepsilon(x)\nabla u^\varepsilon\right) - \dfrac{1}{\varepsilon}\left(b^\varepsilon(x), \nabla u^\varepsilon\right) = \dfrac{1}{\varepsilon} f(x_1), & x \in G_\varepsilon, \\ B^\varepsilon u^\varepsilon \equiv \dfrac{\partial u^\varepsilon}{\partial n_{a^\varepsilon}} = g(x_1), & x \in \varGamma_\varepsilon, \\ u^\varepsilon(-1, x') = \varphi^-\left(\dfrac{x'}{\varepsilon}\right), \quad u^\varepsilon(1, x') = \varphi^+\left(\dfrac{x'}{\varepsilon}\right), & x' \in \varepsilon Q, \end{cases} \tag{26.1}$$

where the matrix-valued function $a^\varepsilon(x)$ and the vector field $b^\varepsilon(x)$ are given by $a^\varepsilon(x) = a(x/\varepsilon)$, $\quad b^\varepsilon(x) = b(x/\varepsilon)$, and $\varepsilon > 0$ is a small parameter. In (26.1) $(\cdot,\cdot)$ stands for the standard scalar product in $\mathbb{R}^d$; $\partial u^\varepsilon/\partial n_{a^\varepsilon} = (a^\varepsilon \nabla u^\varepsilon, n)$ is the co-normal derivative of u^ε, and n is an external unit normal. Throughout the chapter we denote

$$\mathbb{G} = (-\infty, +\infty) \times Q, \quad \varGamma = (-\infty, +\infty) \times \partial Q;$$

$$G_\alpha^\beta = (\alpha, \beta) \times Q, \quad -\infty \le \alpha \le \beta \le +\infty.$$

We suppose the following conditions to hold:

(H1) The coefficients $a_{ij}(y) \in C^{1,\alpha}(\mathbb{G})$ and $b_j(y) \in C^\alpha(\mathbb{G})$ are periodic outside some compact set $K \Subset G_{-1}^1$. More precisely,

$$a_{ij}(y) = \begin{cases} a_{ij}^+(y), & y_1 > 1, \\ \tilde{a}_{ij}(y), & |y_1| \le 1, \\ a_{ij}^-(y), & y_1 < -1; \end{cases} \qquad b(y) = \begin{cases} b_j^+(y), & y_1 > 1, \\ \tilde{b}_j(y), & |y_1| \le 1, \\ b_j^-(y), & y_1 < -1; \end{cases}$$

where $a^\pm(y)$ and $b^\pm(y)$ are periodic in y_1. Without loss of generality, we assume that the period is equal to 1.

(H2) The matrices $a^\pm(y)$ are symmetric.

(H3) We assume that $a^{\pm}(y)$ satisfy the uniform ellipticity condition; that is, there exists a positive constant Λ such that, for almost all $x \in \mathbb{R}^d$,

$$\Lambda\,|\xi|^2 \le \sum_{i,j=1}^{d} a_{ij}^{\pm}(y)\,\xi_i\,\xi_j, \quad \forall \xi \in \mathbb{R}^d. \tag{26.2}$$

(H4) $\varphi^{\pm}(y') \in H^{1/2}(Q)$.

(H5) Functions $f(x_1)$ and $g(x_1)$ are supposed to be smooth, namely, $f(x_1) \in C^2(G_\varepsilon)$ and $g(x_1) \in C^2(\Gamma_\varepsilon)$.

The goal of this work is to study the asymptotic behavior of $u^\varepsilon(x)$, as $\varepsilon \to 0$. As was noted in the Introduction, in contrast to the case of an operator in divergence form, the situation turns out to depend crucially on the signs of the effective fluxes $\bar{b}_1^{\pm}$, the constants which are defined in terms of the kernel of the adjoint periodic operators and coefficients of the equation. When constructing boundary layer functions, we consider only one case: $\bar{b}_1^{+} < 0$, $\bar{b}_1^{-} > 0$.

26.3 Formal Asymptotic Expansion

In the sequel we use the following notation:

$$G_\varepsilon^{+} = \{x = (x_1, x') \in G_\varepsilon : x_1 > \varepsilon\}, \quad G_\varepsilon^{-} = \{x = (x_1, x') \in G_\varepsilon : x_1 < -\varepsilon\};$$
$$A_y^{\pm} v \equiv -\mathrm{div_y}\,(a^{\pm}(y)\nabla_y\, v) - (b^{\pm}(y), \nabla_y v), \quad y \in Y;$$
$$B_y^{\pm} v \equiv \frac{\partial v}{\partial n_{a^{\pm}}} = \sum_{i,j=1}^{d} a_{ij}^{\pm}(y)\,\partial_{y_j} v\, n_i, \quad y \in Y,$$

where $Y = \mathfrak{S}_1 \times Q$, with $\mathfrak{S}_1$ a unit circle, denotes the cell of periodicity. In what follows we identify y_1-periodic functions with functions defined on Y. Notice that $\partial Y = \mathfrak{S}_1 \times \partial Q$.

In each half-cylinder G_ε^{+} and G_ε^{-} the inner asymptotic expansion of a solution to equation (26.1) has the form (see, for example, [BaPa89], [BLP78])

$$\begin{aligned} u_\infty^{\pm} &= v_0^{\pm}(x_1) + \varepsilon\big[N_1^{\pm}\big(\frac{x}{\varepsilon}\big)\,(v_0^{\pm})'(x_1) + v_1^{\pm}(x_1) + q_1^{\pm}\big(\frac{x}{\varepsilon}\big)g(x_1)\big] \\ &+ \varepsilon^2\big[N_2^{\pm}\big(\frac{x}{\varepsilon}\big)\,(v_0^{\pm})''(x_1) + N_1^{\pm}\big(\frac{x}{\varepsilon}\big)\,(v_1^{\pm})'(x_1) + v_2^{\pm}(x_1) + q_2^{\pm}\big(\frac{x}{\varepsilon}\big)g'(x_1)\big]. \end{aligned} \tag{26.3}$$

The leading term of the asymptotics, $v_0^{\pm}$, satisfies a first order ordinary differential equation

$$\bar{b}_1^{\pm}(v_0^{\pm})'(x_1) = f(x_1) + g(x_1)\int\limits_{\partial Y} p^{\pm}(y)d\sigma_y, \tag{26.4}$$

where

$$\bar{b}_1^{\pm} = \int\limits_Y \left(a_{i1}^{\pm}(y)\partial_{y_i} p^{\pm}(y) - b_1^{\pm}(y) p^{\pm}(y)\right) dy$$

is called the effective axial drift; and $p^{\pm}(y)$ belong to the kernels of adjoint periodic operators defined on Y:

$$\begin{cases} -\mathrm{div}\,(a^{\pm}(y)\nabla\, p^{\pm}) + \mathrm{div}\,(b^{\pm}\, p^{\pm}) = 0, & y \in Y, \\ \dfrac{\partial p^{\pm}}{\partial n_{a^{\pm}}} - (b^{\pm}, n)\, p^{\pm} = 0, & y \in \partial Y. \end{cases}$$

Throughout the chapter we will assume that

(H6) $\bar{b}_1^- > 0$ *and* $\bar{b}_1^+ < 0$.

Notice that since $f(x_1), g(x_1) \in C^2([-1,1])$, then $v_0^+(x_1) \in C^3(\varepsilon, 1)$, $v_0^-(x_1) \in C^3(-1, -\varepsilon)$.

One can see that necessarily the functions $N_1^{\pm}$ and $q_1^{\pm}$ satisfy the problems

$$\begin{cases} A_y^{\pm}\, N_1^{\pm} = \partial_{y_i} a_{i1}^{\pm} + b_1^{\pm} + \bar{b}_1^{\pm}, & y \in Y, \\ B_y^{\pm}\, N_1^{\pm} = -a_{i1}^{\pm}\, n_i, & y \in \partial Y; \end{cases} \qquad \begin{cases} A_y^{\pm}\, q_1^{\pm} = -\displaystyle\int_{\partial Y} p^{\pm} d\sigma_y, & y \in Y, \\ B_y^{\pm}\, q_1^{\pm} = 1, & y \in \partial Y. \end{cases} \tag{26.5}$$

Obviously, by the definition of $\bar{b}_1^{\pm}$, the compatibility conditions for (26.5) are satisfied; thus, these problems are uniquely (up to an additive constant) solvable. Since we assumed that $a_{ij}(y) \in C^{1,\alpha}(\overline{\mathbb{G}})$ and $b_j(y) \in C^{\alpha}(\overline{\mathbb{G}})$, then $N_1^{\pm}(y)$ and $q_1^{\pm}(y)$ belong to $C^{2,\alpha}(\overline{Y})$ (see, for example, [GiTr98], [LaUr68]).

The equation for $v_1^{\pm}$ reads

$$\bar{b}_1^{\pm}\,(v_1^{\pm})'(x_1) = h_2^{\pm}\,(v_0^{\pm})''(x_1) + \overline{q_1^{\pm}}\, g'(x_1), \tag{26.6}$$

where $h_2^{\pm}$ and $\overline{q_1^{\pm}}$ are constants given by the following expressions:

$$h_2^{\pm} = \int_Y \left(a_{11}^{\pm}\, p^{\pm} - a_{i1}^{\pm}\, N_1^{\pm}(y)\partial_{y_i} p^{\pm} + b_1^{\pm}\, N_1^{\pm}\, p^{\pm} + a_{1j}^{\pm}\, \partial_{y_j} N_1^{\pm}\, p^{\pm}\right) dy;$$
$$\overline{q_1^{\pm}} = \int_Y \left(- a_{i1}^{\pm}\, q_1^{\pm} \partial_{y_i} p^{\pm} + b_1^{\pm}\, q_1^{\pm} p^{\pm} + a_{1j}^{\pm}\, \partial_{y_j} q_1^{\pm}\, p^{\pm}\right) dy.$$

Let us note that $v_1^{\pm}(x_1)$, as a solution of (26.6), has continuous derivatives in $\overline{Y}$ up to the second order.

One can see that $N_2^{\pm}$ and $q_2^{\pm}$ satisfy the problems

$$\begin{cases} A_y^{\pm}\, N_2^{\pm} = a_{11}^{\pm} + \partial_{y_i}(a_{i1}^{\pm}\, N_1^{\pm}) + b_1^{\pm}\, N_1^{\pm} + a_{1j}^{\pm} \partial_{y_j} N_1^{\pm} - h_2^{\pm}, & y \in Y, \\ B_y^{\pm}\, N_2^{\pm} = -a_{i1}^{\pm}\, N_1^{\pm}\, n_i, & y \in \partial Y; \end{cases} \tag{26.7}$$

$$\begin{cases} A_y^{\pm}\, q_2^{\pm} = \partial_{y_i}(a_{i1}^{\pm}\, q_1^{\pm}) + b_1^{\pm}\, q_1^{\pm} + a_{1j}^{\pm}\, \partial_{y_j} q_1^{\pm} - \overline{q_1^{\pm}}, & y \in Y, \\ B_y^{\pm}\, q_1^{\pm} = -a_{i1}^{\pm}\, q_1^{\pm}\, n_i, & y \in \partial Y. \end{cases} \tag{26.8}$$

The compatibility conditions are satisfied and problems (26.7)–(26.8) are uniquely solvable. The smoothness of the coefficients and the properties of the functions $N_1^{\pm}$, $q_1^{\pm}$ imply that $N_2^{\pm}(y), q_2^{\pm}(y) \in C^{2,\alpha}(\overline{Y})$.

The equation for $v_2^{\pm}(x_1)$ is the following:

$$\bar{b}_1^{\pm}\,(v_2^{\pm})'(x_1) = h_3^{\pm}\,(v_0^{\pm})^{(3)}(x_1) + h_2^{\pm}\,(v_1^{\pm})''(x_1) + \overline{q_2^{\pm}}\,g''(x_1), \tag{26.9}$$

where

$$h_3^{\pm} = \int_Y \big(a_{11}^{\pm}N_1^{\pm}p^{\pm} - a_{i1}^{\pm}N_2^{\pm}\partial_{y_i}p^{\pm} + b_1^{\pm}N_2^{\pm}p^{\pm} + a_{1j}^{\pm}\partial_{y_j}N_2^{\pm}p^{\pm}\big)\,dy;$$
$$\overline{q_2^{\pm}} = \int_Y \big(a_{11}^{\pm}\,q_1^{\pm}p^{\pm} - a_{i1}^{\pm}q_2^{\pm}\partial_{y_i}p^{\pm} + b_1^{\pm}q_2^{\pm}p^{\pm} + a_{1j}^{\pm}\partial_{y_j}q_2^{\pm}p^{\pm}\big)\,dy.$$

The function $v_2^{\pm}$ as a solution of (26.9) is a $C^1(\overline{Y})$ function.

Note that the infinite number of terms in series (26.3) can be constructed. Interested readers can find in [Pa05] the description of the general method for such a construction together with some applications and examples.

26.4 Boundary Layers Near the Rod Ends

The asymptotic series (26.3) does not satisfy the boundary conditions on the bases of the rod, which is why we introduce the boundary layer functions in the neighborhoods of $S_{\pm 1} = \{x \in G_\varepsilon : x_1 = \pm 1,\ x' \in \varepsilon Q\}$:

$$\begin{aligned} v_{bl}^{\pm}(x) \equiv \big[w_0^{\pm}\big(\frac{x_1 \mp 1}{\varepsilon}, \frac{x'}{\varepsilon}\big) - \hat{w}_0^{\pm}\big] + \varepsilon\big[w_1^{\pm}\big(\frac{x_1 \mp 1}{\varepsilon}, \frac{x'}{\varepsilon}\big) - \hat{w}_1^{\pm}\big] \\ + \varepsilon^2\,\big[w_2^{\pm}\big(\frac{x_1 \mp 1}{\varepsilon}, \frac{x'}{\varepsilon}\big) - \hat{w}_2^{\pm}\big]. \end{aligned} \tag{26.10}$$

Here $w_0^{\pm}(y)$ are the solutions of homogeneous problems in semi-infinite cylinders $G_{-\infty}^{0}$ and $G_0^{+\infty}$, respectively,

$$\begin{cases} A_y^{+}w_0^{+}(y) = 0, \quad y \in G_{-\infty}^{0}, \\ B_y^{+}w_0^{+} = 0, \quad y \in \Gamma_{-\infty}^{0}, \\ w_0^{+}(0,y') = \varphi^{+}(y'), \end{cases} \qquad \begin{cases} A_y^{-}w_0^{-}(y) = 0, \quad y \in G_0^{+\infty}, \\ B_y^{-}w_0^{-} = 0, \quad y \in \Gamma_0^{+\infty}, \\ w_0^{-}(0,y') = \varphi^{-}(y'). \end{cases} \tag{26.11}$$

As was proved in [PaPi09] (see Theorem 5.1), under assumptions (**H1**)−(**H6**), problems (26.11) possess unique solutions stabilizing to constants $\hat{w}_0^{\pm}$ at an exponential rate, as $y_1 \to \mp\infty$. As boundary conditions for $v_0^{\pm}$ we choose $v_0^{\pm}(\pm 1) = \hat{w}_0^{\pm}$.

The functions $w_1^{\pm}$ satisfy the following problems:

$$\begin{cases} A_y^{+}w_1^{+}(y) = 0, \quad y \in G_{-\infty}^{0}, \\ B_y^{+}w_1^{+} = 0, \quad y \in \Gamma_{-\infty}^{0}, \\ w_1^{+}(0,y') = -N_1^{+}(\delta,y')\,(v_0^{+})'(1) \\ -q_1^{+}(\delta,y')\,g(1), \end{cases} \qquad \begin{cases} A_y^{-}w_1^{-}(y) = 0, \quad y \in G_0^{+\infty}, \\ B_y^{-}w_1^{-} = 0, \quad y \in \Gamma_0^{+\infty}, \\ w_1^{-}(0,y') = -N_1^{-}(-\delta,y')\,(v_0^{-})'(-1) \\ -q_1^{-}(-\delta,y')\,g(-1), \end{cases}$$

for some fixed $\delta \in [0,1)$ (δ is a fractional part of ε^{-1}). Taking into account that $\bar{b}_1^+ < 0$, $\bar{b}_1^- > 0$, one can see that $w_1^\pm$ stabilize to uniquely defined constants which we denote by $\hat{w}_1^\pm$ (see [PaPi09]). Then we take the constant $\hat{w}_1^\pm$ as boundary conditions for $v_1^\pm(x_1)$ as $x_1 = \pm 1$: $v_1^\pm(\pm 1) = \hat{w}_1^\pm$.

Turning back to (26.10), $w_2^\pm$ solve the problems

$$\begin{cases} A_y^\pm w_2^\pm = 0, \quad y \in G_{-\infty}^0 \ (y \in G_0^{+\infty}), \\ B_y^\pm w_2^\pm = 0, \quad y \in \Gamma_{-\infty}^0 \ (y \in \Gamma_0^{+\infty}), \\ w_2^\pm(0,y') = -N_2^\pm(\pm\delta, y')\,(v_0^\pm)''(\pm 1) \\ -N_1^\pm(\pm\delta, y')\,(v_1^\pm)'(\pm 1) - q_2^\pm(\pm\delta, y')\,g'(\pm 1). \end{cases} \tag{26.12}$$

$w_2^\pm$ tend to constants $\hat{w}_2^\pm$, as $y_1 \to \mp\infty$. As before, the existence and uniqueness of solutions and the property of the exponential stabilization to constants are ensured by Theorem 5.1 in [PaPi09]. Now we can choose a boundary condition for the functions $v_2^\pm$ as $x_1 = \pm 1$: $v_2^\pm(1) = \hat{w}_2^\pm$.

26.5 Boundary Layer in the Middle of the Rod

Before constructing the boundary layer functions in the middle of the rod, let us extend $v_0^+(x_1)$ (keeping the same notation) to $(-\infty, \varepsilon)$ as a solution of equation (26.4) satisfying the boundary condition $v_0^+(1) = \hat{w}_0^+$. In the same way we can extend v_1^+, v_2^+ to $(-\infty, \varepsilon)$, and v_0^-, v_1^-, v_2^- to $(-\varepsilon, +\infty)$ as solutions to corresponding ordinary differential equations. Periodic in y_1 functions $N_k^\pm$ and $q_k^\pm$, $k = 1,2,3$, we regard as defined everywhere in $\mathbb{G} = \mathbb{R} \times Q$.

Obviously, it suffices to match the formal asymptotic series u_∞^+, defined by (26.3) in G_∞^+, with zero in the vicinity of $S_0^\varepsilon = \{x \in G_\varepsilon : x_1 = 0\}$. Then, in the same way we can match u_∞^- with zero, and, summing up the obtained expressions, arrive at the final boundary layer corrector in the neighborhood of S_0^ε. In order to do this, we are looking for a "corrected" solution in the form

$$\begin{aligned} v_\varepsilon^\pm(x) &= \chi_0^\pm(y)\,v_0^\pm(x_1) + \varepsilon\,N_1^\pm(y)\,\phi^\pm(y)\,(v_0^\pm)'(x_1) + \varepsilon\,\chi_{1,1}^\pm(y)\,(v_0^\pm)'(x_1) \\ &+ \varepsilon\,q_1^\pm(y)\,\phi^\pm(y) g(x_1) + \varepsilon\,\chi_{1,2}^\pm\,g(x_1) + \varepsilon\,\chi_1^\pm(y)\,v_1^\pm(x_1) \\ &+ \varepsilon^2\,N_2^\pm(y)\,\phi^\pm(y)\,(v_0^\pm)''(x_1) + \varepsilon^2\,\chi_{2,1}^\pm(y)\,(v_0^\pm)''(x_1) \\ &+ \varepsilon^2\,N_1^\pm(y)\,\phi^\pm(y)\,(v_1^\pm)'(x_1) + \varepsilon^2\,\chi_{2,2}^\pm(y)\,(v_1^\pm)'(x_1) \\ &+ \varepsilon^2\,q_2^\pm(y)\,\phi^\pm(y) g'(x_1) + \varepsilon^2\,\chi_{2,3}^\pm(y)\,g'(x_1) + \varepsilon^2\,\chi_2^\pm(y)\,v_2^\pm(x_1), \quad y = x/\varepsilon, \end{aligned} \tag{26.13}$$

where the functions $\chi_1^\pm(y)$, $\chi_{1,1}^\pm(y)$, $\chi_{1,2}^\pm(y)$, $\chi_{2,1}^\pm(y)$, $\chi_{2,2}^\pm(y)$, $\chi_{2,3}^\pm(y)$, and $\chi_2^\pm(y)$ are to be determined; $\phi^+(y) = \phi^+(y_1)$ is a smooth cut-off function such that $\phi^+(y) = 0$ if $y_1 < -1$ and $\phi^+(y) = 1$ if $y_1 > 1$, $\phi^- = 1 - \phi^+$.

Substituting (26.13) into (26.1) and collecting power-like terms related to different powers of ε, one gets equations for the unknown functions. Due to lack of space, we do not produce the calculations here.

$$\begin{cases} A_y\,\chi_m^{\pm} = 0, & y \in \mathbb{G}, \\ B_y\,\chi_m^{\pm} = 0, \quad y \in \varGamma, & m = 0,1,2. \end{cases} \tag{26.14}$$

$$\begin{cases} A_y \chi_{1,1}^{\pm} = -A_y(N_1^{\pm}(y)\phi^{\pm}(y)) + a_{1j}(y)\partial_{y_j}\chi_0^{\pm}(y) \\ \qquad\qquad + \partial_{y_i}\big(a_{i1}\chi_0^{\pm}(y)\big) + b_1(y)\chi_0^{\pm}(y) - \bar{b}_1^{\pm}\,\phi^{\pm}(y), \quad y \in \mathbb{G}; \\ B_y\chi_{1,1}^{\pm} = -a_{i1}\,\chi_0^{\pm}\,n_i - a_{ij}\,\partial_{y_j}(N_1^{\pm}\,\phi^{\pm})\,n_i, \qquad y \in \varGamma; \end{cases} \tag{26.15}$$

$$\begin{cases} A_y\chi_{1,2}^{\pm} = -A_y\big(q_1^{\pm}(y)\phi^{\pm}(y)\big) - \phi^{\pm}(y)\displaystyle\int_{\partial Y} p^{\pm}(y)\,d\sigma_y, \quad y \in \mathbb{G}, \\ B_y\chi_{1,2}^{\pm} = -a_{ij}\,\partial_{y_j}\big(q_1^{\pm}(y)\,\phi^{\pm}(y)\big)\,n_i + \phi^{\pm}(y), \qquad y \in \varGamma; \end{cases} \tag{26.16}$$

Problems (26.14)–(26.16), stated in the infinite cylinder $\mathbb{G}$, were derived by formal calculations which, of course, do not imply the solvability of these problems. Theorem 2, proved in Section 26.7, guarantees the existence of solutions to problems (26.14)–(26.16) in proper classes and, moreover, gives an additional qualitative information about the solutions. Indeed, we can choose $\chi_m^{\pm}$, $m = 0,1,2$, such that

$$\begin{aligned} \chi_m^{+} \underset{y_1\to+\infty}{\longrightarrow} 1, \quad \chi_m^{+} \underset{y_1\to-\infty}{\longrightarrow} 0; \\ \chi_m^{-} \underset{y_1\to+\infty}{\longrightarrow} 0, \quad \chi_m^{-} \underset{y_1\to-\infty}{\longrightarrow} 1, \quad m = 0,1,2. \end{aligned} \tag{26.17}$$

Such a choice of $\chi_0^{\pm}$ and definitions of $N_1^{\pm}(y)$ and $\phi^{\pm}(y)$ ensure the existence of solutions $\chi_{1,1}^{\pm}$ of problem (26.15), which stabilize to the constants at infinity. For the functions $\chi_{1,1}^{\pm}$ we assign zeros at infinity: $\chi_{1,1}^{\pm} \to 0, \quad y_1 \to \pm\infty.$

Similarly, taking into account (26.5) and the definition of $\phi^{\pm}$, one can see that problems (26.16) are solvable. We also choose zeros as constants at infinity for $\chi_{1,2}^{\pm}$: $\chi_{1,2}^{\pm} \to 0, \quad y_1 \to \pm\infty.$

In much the same way, we see that there exist $\chi_{2,1}^{\pm}$, $\chi_{2,2}^{\pm}$, $\chi_{2,3}^{\pm}$ stabilizing to zero, as $y_1 \to \pm\infty$, which solve the following problems:

$$\begin{cases} A_y\,\chi_{2,1}^{+} = -A_y(N_2^{+}\,\phi^{+}) + a_{11}\,\chi_0^{+} + a_{1j}\,\partial_{y_j}(N_1^{+}\,\phi^{+}) + \partial_{y_i}(a_{i1}\,N_1^{+}\,\phi^{+}) \\ + b_1\,N_1^{+}\,\phi^{+} + a_{1j}\,\partial_{y_j}\chi_{1,1}^{+} + \partial_{y_i}(a_{i1}\,\chi_{1,1}^{+}) + b_1\,\chi_{1,1}^{+} - h_2^{+}\,\phi^{+}, \quad y \in \mathbb{G}, \\ B_y\,\chi_{2,1}^{+} = -B_y\,(N_2^{+}\,\phi^{+}) - a_{i1}\,n_i\,\chi_{1,1}^{+} - a_{i1}\,n_i\,N_1^{+}\,\phi^{+}, \quad y \in \varGamma; \end{cases} \tag{26.18}$$

$$\begin{cases} A_y\,\chi_{2,2}^{+} = -A_y(N_1^{+}\,\phi^{+}) + a_{1j}\,\partial_{y_j}\chi_1^{+} \\ \qquad\qquad + \partial_{y_i}(a_{i1}\chi_1^{+}) + b_1\,\chi_1^{+} - \bar{b}_1^{+}\,\phi^{+}, \quad y \in \mathbb{G}, \\ B_y\,\chi_{22}^{+} = -B_y(N_1^{+}\,\phi^{+}) - a_{i1}\,n_i\,\chi_1^{+}, \qquad y \in \varGamma; \end{cases} \tag{26.19}$$

$$\begin{cases} A_y\,\chi_{2,3}^{+} = -A_y(q_2^{+}\,\phi^{+}) + a_{1j}\,\partial_{y_j}(q_1^{+}\,\phi^{+}) + \partial_{y_i}(a_{i1}\,q_1^{+}\,\phi^{+}) + b_1\,q_1^{+}\,\phi^{+} \\ + a_{1j}\,\partial_{y_j}\chi_{1,2}^{+} + \partial_{y_i}(a_{i1}\,\chi_{1,2}^{+}) + b_1\,\chi_{1,2}^{+} - \overline{q_1^{+}}\,\phi^{+}, \quad y \in \mathbb{G}, \\ B_y\,\chi_{2,3}^{+} = -B_y\,(q_2^{+}\,\phi^{+}) - a_{i1}\,n_i\,\chi_{1,2}^{+} - a_{i1}\,n_i\,q_1^{+}\,\phi^{+}, \quad y \in \varGamma. \end{cases} \tag{26.20}$$

Finally, taking into account the constructed inner formal asymptotic expansion and boundary layer correctors in the neighborhoods of $S_{\pm 1}$ and S_0, we arrive at the asymptotic solution of problem (26.1):

$$u^{\varepsilon}_{\infty}(x) \equiv v^{+}_{\varepsilon}(x) + v^{+}_{bl}(x) + v^{-}_{\varepsilon}(x) + v^{-}_{bl}(x), \tag{26.21}$$

where v^{+}_{ε}, v^{-}_{ε}, and $v^{\pm}_{bl}$ are defined by (26.13) and (26.10).

Remark 1. Adding the boundary layer functions $v^{\pm}_{bl}$ to the inner expansions $u^{\pm}_{\infty}$ makes it possible to satisfy the boundary conditions on the bases of the rod G_ε with an accuracy up to the third order in ε. Representing (26.21) as the sum of the inner expansions and the boundary layer functions

$$\begin{aligned} u^{\varepsilon}_{\infty} &= u^{+}_{\infty}(x) + (v^{+}_{\varepsilon}(x) - u^{+}_{\infty}(x)) + v^{+}_{bl}(x) \\ &+ u^{-}_{\infty}(x) + (v^{-}_{\varepsilon}(x) - u^{-}_{\infty}(x)) + v^{-}_{bl}(x), \end{aligned}$$

we make $(v^{\pm}_{\varepsilon} - u^{\pm}_{\infty})$ exponentially small (but not vanishing) on $S^{\varepsilon}_{\pm}$, as well as v^{+}_{bl} on S^{ε}_{-1} and v^{-}_{bl} on S^{ε}_{+1}. In order to satisfy exactly the boundary conditions, one can replace (26.21) with

$$\begin{aligned} \tilde{u}^{\varepsilon}_{\infty} &= u^{+}_{\infty}(x) + (v^{+}_{\varepsilon}(x) - u^{+}_{\infty}(x))\, \phi_1(x) + v^{+}_{bl}(x)\, \phi^{+}_1(x) \\ &+ u^{-}_{\infty}(x) + (v^{-}_{\varepsilon}(x) - u^{-}_{\infty}(x))\, \phi_1(x) + v^{-}_{bl}(x)\, \phi^{-}_1(x), \end{aligned}$$

where $\phi_1(x) = 1$ if $|x_1| < 1/3$ and $\phi_1(x) = 0$ otherwise;

$$\phi^{+}_1(x) = \begin{cases} 1, & x_1 > 2/3, \\ 0, & x_1 < 1/3. \end{cases} \qquad \phi^{-}_1(x) = \begin{cases} 1, & x_1 < -2/3, \\ 0, & x_1 > -1/3. \end{cases}$$

Substituting $\tilde{u}^{\varepsilon}_{\infty}$ into (26.1), it is straightforward to check that the presence of the cut-off functions results in the appearance of additional exponentially small (with respect to ε^{-1}) terms on the right-hand side. Later on we will prove a priori estimates (26.23) and (26.24) which ensure that the exponentially small perturbation of the right-hand side leads to the exponentially small perturbation of the solution, and, thus, is negligible in any polynomial in ε expansion. To simplify the notation, we deal with (26.21) neglecting the discrepancies on $S^{\varepsilon}_{\pm 1}$ which are exponentially small with respect to ε^{-1}.

26.6 Justification of the Procedure

Theorem 1. *Let the conditions* (**H1**)–(**H6**) *hold true. Then the approximate solution u^{ε}_{∞} given by formula* (26.21) *satisfies the estimates*

$$\begin{aligned} \|\nabla u^{\varepsilon}_{\infty} - \nabla u^{\varepsilon}\|_{L^2(G_\varepsilon)} &\le C\,\varepsilon^{3/2}\,\varepsilon^{(d-1)/2}, \\ \|u^{\varepsilon}_{\infty} - u^{\varepsilon}\|_{L^2(G_\varepsilon)} &\le C\,\varepsilon^{3/2}\,\varepsilon^{(d-1)/2}, \end{aligned} \tag{26.22}$$

where $u^{\varepsilon}(x)$ is the exact solution to problem (26.1).

Proof. First we obtain an a priori estimate for a solution to the problem

$$\begin{cases} A^\varepsilon u^\varepsilon = f^\varepsilon(x), & x \in G_\varepsilon, \\ B^\varepsilon u^\varepsilon = g^\varepsilon(x), & x \in \Gamma_\varepsilon, \\ u^\varepsilon(\pm 1, x') = 0, & x' \in \varepsilon Q \end{cases}$$

in terms of $f^\varepsilon(x)$ and $g^\varepsilon(x)$ (for the moment we do not specify the particular structure of these functions). While proving Theorem 2 in Section 26.7, we will show that the following estimates hold true:

$$\|\nabla u^\varepsilon\|_{L^2(G_\varepsilon)} \le C\sqrt{\varepsilon}\,\|f^\varepsilon\|_{L^2(G_\varepsilon)} + C\sqrt{\varepsilon}\,\|g^\varepsilon\|_{L^2(\Gamma_\varepsilon)}. \tag{26.23}$$

Making use of the Friedrichs inequality for the function u^ε in G_ε, we obtain

$$\|u^\varepsilon\|_{L^2(G_\varepsilon)} \le C\sqrt{\varepsilon}\,\|f^\varepsilon\|_{L^2(G_\varepsilon)} + C\sqrt{\varepsilon}\,\|g^\varepsilon\|_{L^2(\Gamma_\varepsilon)}. \tag{26.24}$$

Estimation of the $L^2(G_\varepsilon)$-norm of $A^\varepsilon\big((v_\varepsilon^+ + v_{bl}^+) + (v_\varepsilon^- + v_{bl}^-) - u^\varepsilon\big)$ and the $L^2(\Gamma_\varepsilon)$-norm of $B^\varepsilon\big((v_\varepsilon^+ + v_{bl}^+) + (v_\varepsilon^- + v_{bl}^-) - u^\varepsilon\big)$ will complete the justification procedure. Due to lack of space, we have to drop these estimates and leave them to the reader.

$$\big\|A^\varepsilon\big((v_\varepsilon^+ + v_{bl}^+) + (v_\varepsilon^- + v_{bl}^-) - u^\varepsilon\big)\big\|_{L^2(G_\varepsilon)} \le C\,\varepsilon\,\varepsilon^{(d-1)/2};$$

$$\big\|B^\varepsilon\big((v_\varepsilon^+ + v_{bl}^+) + (v_\varepsilon^- + v_{bl}^-) - u^\varepsilon\big)\big\|_{L^2(\Gamma_\varepsilon)} \le C\,\varepsilon^2\,\varepsilon^{(d-2)/2}; \tag{26.25}$$

Taking into account (26.23)–(26.25) we get (26.22).

Remark 2. The estimates (26.23)–(26.24) imply that we can take $f(x_1) \in L^2(G_\varepsilon)$ and $g(x_1) \in L^2(\Gamma_\varepsilon)$.

26.7 Existence of a Solution in an Infinite Cylinder

We consider the following boundary value problem:

$$\begin{cases} A_\# u \equiv -\mathrm{div}\,(a(x)\,\nabla u(x)) - (b(x), \nabla u(x)) = f(x), & x \in \mathbb{G}, \\ B_\# u \equiv \dfrac{\partial u}{\partial n_a} = g(x), & x \in \Gamma. \end{cases} \tag{26.26}$$

We assume that
$(\mathbf{H5})'$ The functions $f \in C(\bar{\mathbb{G}})$ and $g \in C(\Gamma)$ are such that

$$\|f\|_{L^2(G_n^{n+1})} \le Ce^{-\gamma_1 n}, \quad \|g\|_{\Gamma^2(\Gamma_n^{n+1})} \le Ce^{-\gamma_1 n}, \quad \gamma_1 > 0, n \in \mathbb{R}.$$

The goal of this section is to show that in the case $\bar{b}_1^+ < 0$, $\bar{b}_1^- > 0$, problem (26.26) possesses a bounded (in a proper sense) solution, which stabilizes to constants, as $|x_1| \to \infty$.

Definition 1. *A weak solution $u(x)$ of problem* (26.26) *is said to be bounded if*

$$\|u\|_{L^2(G_n^{n+1})} \le C,$$

with a constant C independent of n.

The following theorem contains the main result of the section.

Theorem 2. *Let conditions* $(\mathbf{H1})-(\mathbf{H3}), (\mathbf{H5})', (\mathbf{H6})$ *be fulfilled. Then for any constants K_∞^+ and K_∞^- there exists a bounded solution $u(x)$ of problem* (26.26) *such that it converges at the exponential rate to these constants, as $x_1 \to \pm\infty$,*

$$\|u - K_\infty^-\|_{L^2(G_{-\infty}^{-n})} \le C\,(1 + K_\infty^-)\, e^{-\gamma\, n},$$

$$\|u - K_\infty^+\|_{L^2(G_n^{+\infty})} \le C\,(1 + K_\infty^+)\, e^{-\gamma\, n}, \quad \gamma > 0,$$

and the following estimates hold:

$$\|u\|_{L^2(G_n^{n+1})} \le C\,\big(\|(1+\sqrt{|x_1|})\, f\|_{L^2(\mathbb{G})} + \|(1+\sqrt{|x_1|})\, g\|_{L^2(\Gamma)}\big);$$
$$\|\nabla\, u\|_{L^2(\mathbb{G})} \le C\,\big(\|(1+\sqrt{|x_1|})\, f\|_{L^2(\mathbb{G})} + \|(1+\sqrt{|x_1|})\, g\|_{L^2(\Gamma)}\big).$$

Proof. Let us consider the following sequence of auxiliary boundary value problems in a growing family of finite cylinders:

$$\begin{cases} A_\#\, u_k = f(x), & x \in G_{-k}^k, \\ B_\#\, u_k = g(x), & x \in \Gamma_{-k}^k, \\ u_k(-k, x') = u_k(k, x') = 0, & x' \in Q. \end{cases} \tag{26.27}$$

Without loss of generality, we assume that $f(x) > 0$ and $g(x) > 0$. Moreover, we assume that the functions f and g are equal to zero in the half-cylinder $G_{-\infty}^0$; that is, $\operatorname{supp} f, \operatorname{supp} g \subset G_0^{+\infty}$. The case when the supports of f and g belong to $G_{-\infty}^0$ can be considered similarly. Due to the regularity assumptions $(\mathbf{H1}), (\mathbf{H5})'$, the maximum principle and the boundary point lemma are valid (see, e.g., [GiTr98]), and, consequently, a negative minimum cannot be attended in the internal part of G_{-k}^k and its lateral boundary; that is, $u_k \ge 0$ in G_{-k}^k.

In the cylinder G_{-k}^{-1} the function $u_k(x)$ is a solution of a homogeneous equation. Since $u_k(-k, x') = 0$ and $\bar{b}_1^- > 0$, we have the following estimate:

$$u_k(x) \le \|u_k\|_{L^\infty(S_{-1})}\, e^{\gamma\, x_1}, \quad x \in G_{-k}^{-1},\ \gamma > 0.$$

The proof of this fact can be found in [PaPi09], Section 5, Theorem 5.5.

For the nonnegative function $u_k(x)$, the Harnack inequality is valid in the fixed domain G_{-1}^0 with a constant α which depends only on $d, |Q|$, and Λ; that is,

$$u_k(x) \le \alpha \min_{G_{-1}^0} u_k(x)\, e^{\gamma\, x_1}.$$

Obviously, there exists $\xi > 1$, independent of k, such that

$$u_k(-\xi, x') < \frac{1}{2} \min_{G^0_{-1}} u_k(x). \tag{26.28}$$

In $G^k_{-\xi}$, due to the linearity of the problem, we can represent u_k as a sum $v_k + w_k$, where v_k is a solution of the homogeneous equation with nonzero Dirichlet boundary condition $v_k(-\xi, x') = u_k(-\xi, x')$; and w_k is a solution of the nonhomogeneous equation with functions f and g on the right-hand side and homogeneous Dirichlet boundary conditions on the bases. In view of the maximum principle and (26.28) we obtain an estimate for $v_k(x)$,

$$v_k(x) \leq \frac{1}{2} \min_{G^0_{-1}} u_k(x), \quad x \in G^k_{-\xi}. \tag{26.29}$$

One can prove (see [PaPi09], Lemma 7.2, estimates (7.10), (7.11)) that the following estimate for w_k holds:

$$\|w_k\|_{L^2(G_N^{N+1})} \leq C\,\|(1+\sqrt{x_1})\,f\|_{L^2(G_0^{+\infty})} + C\,\|(1+\sqrt{x_1})\,g\|_{L^2(\Gamma_0^{+\infty})}. \tag{26.30}$$

In this way, taking into account (26.29) and (26.30), one can see that

$$\min_{G^0_{-1}} u_k(x) \leq \|u_k\|_{L^2(G^0_{-1})} \leq \frac{1}{2} \min_{G^0_{-1}} u_k(x) + \|w_k\|_{L^2(G^0_{-1})}.$$

It follows from the last inequality that

$$\min_{G^0_{-1}} u_k(x) \leq C\,\|(1+\sqrt{x_1})\,f\|_{L^2(G_0^{+\infty})} + C\,\|(1+\sqrt{x_1})\,g\|_{L^2(\Gamma_0^{+\infty})}. \tag{26.31}$$

In view of the Harnack inequality and (26.31), $u_k(-1, x') \leq C$. Then, by the maximum principle,

$$u_k(x) \leq C\,\|(1+\sqrt{x_1})\,f\|_{L^2(\mathbb{G})} + C\,\|(1+\sqrt{x_1})\,g\|_{L^2(\Gamma)}, \quad x \in G^{-1}_{-k}.$$

Combining the last estimate with (26.29)–(26.31) and recalling the relation $u_k = v_k + w_k$, we see that

$$\|u_k\|_{L^2(G_N^{N+1})} \leq C\,\|(1+\sqrt{x_1})\,f\|_{L^2(\mathbb{G})} + C\,\|(1+\sqrt{x_1})\,g\|_{L^2(\Gamma)}, \quad N \in \mathbb{Z},$$

where the constant C does not depend on k. Standard elliptic estimates imply

$$\|\nabla u_k\|_{L^2(G_N^{N+1})} \leq C\,\|(1+\sqrt{x_1})\,f\|_{L^2(\mathbb{G})} + C\,\|(1+\sqrt{x_1})\,g\|_{L^2(\Gamma)}.$$

Thus, up to a subsequence, $u_k(x)$ converges in $H^1_{loc}(\mathbb{G})$ to some function $u(x)$, as $k \to \infty$. Passing to the limit in the integral identity, one can see that $u(x)$ solves problem (26.26). The existence of a bounded solution to problem (26.1) is proved. The result on the exponential stabilization to a constant at $+\infty$ and $-\infty$ of a solution to problem (26.26) follows from the similar results for equations stated in a semi-infinite cylinder (see [PaPi09], Theorem 7.6).

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27

Existence of Extremal Solutions of Singular Functional Cauchy and Cauchy–Nicoletti Problems

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27.1 Introduction

In this chapter we apply fixed point results for mappings in partially ordered spaces presented in ([CaHe00], [HeiLa94]) to derive existence results for the Cauchy problem

$$q(u(t))u'(t) = f(t,u) \text{ for a.e. } t \in J = [0,T],\ u(0) = 0, \tag{27.1}$$

and for the Cauchy–Nicoletti problem

$$q_i(u_i(t))u_i'(t) = f_i(t,u) \text{ for a.e. } t \in J,\ u_i(t_i) = c_i,\ i = 1,...,n, \tag{27.2}$$

where $0 = t_1 < t_2 < \cdots < t_n = T,\ c = (c_1,...,c_n) \in \mathbb{R}^n$. The considered problems include the following special types:

- The differential equations in (27.1) and (27.2) can be singular because $q(0) = 0$ and $q_i(c_i) = 0$ are allowed.
- The right-hand sides of the differential equations in (27.1) and (27.2) depend functionally on the unknown function u.
- The functions q, q_i, f, and f_i may be discontinuous in all their arguments.

As an application, we calculate the least and greatest solutions of (27.1) in the case when $J = [0,1]$,

$$q(y) = \frac{y^2}{1+y^2} \quad \text{and} \quad f(t,u) = \frac{3}{2}\cos(t)\,\overline{\arctan}\left(3D(t) + [\int_0^1 u(t)\,dt]\right),$$

where $[\cdot]$ denotes the greatest integer function and D is the Dirichlet function.

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27.2 Cauchy Problem

Denote $C_+(J) = \{u : J \to \mathbb{R}_+ \mid u \text{ is continuous}\}$, and equip $C_+(J)$ with a pointwise ordering. We shall show that if the functions $q : \mathbb{R}_+ \to \mathbb{R}_+$ and $f : J \times C_+(J) \to \mathbb{R}_+$ satisfy

(h0) $f(\cdot, u)$ is Lebesgue measurable and $f(\cdot, u) \le h \in L^1(J)$ for all $u \in C_+(J)$;
(h1) $0 < \int_0^t f(s,u)\, ds \le \int_0^t f(s,v)\, ds$ whenever $u \le v$ in $C_+(J)$, $t \in (0, T]$;
(h2) $q \in L^1_{loc}(\mathbb{R}_+)$, $\frac{1}{q} \in L^\infty_{loc}(0,\infty)$, and $\int_0^b q(y)\, dy \ge \int_0^T h(s)\, ds$ for some $b > 0$,

then the Cauchy problem (27.1) has least and greatest on $(0, T]$ locally absolutely continuous solutions, and they are increasing with respect to f. We shall convert the Cauchy problem (27.1) to a fixed-point equation $u = Gu$, where the operator G is determined by the following lemma.

Lemma 1. *Let the hypotheses (h0)–(h2) hold. Then the equation*

$$\int_0^{Gu(t)} q(y)\, dy = \int_0^t f(s,u)\, ds, \quad t \in J,\ u \in C_+(J) \tag{27.3}$$

defines a mapping $G : C_+(J) \to C_+(J)$. Moreover, for each $t_0 \in (0, T)$ there exists a positive constant $M(t_0)$ such that

$$0 < Gu(x) \le Gu(t) \le Gu(x) + M(t_0) \int_x^t h(s)\, ds \tag{27.4}$$

whenever $u \in C_+(J)$ and $t_0 \le x \le t \le T$.

Proof. Assume that $u \in C_+(J)$. The hypotheses (h0)–(h2) imply that (27.3) defines a mapping $G : C_+(J) \to C_+(J)$. Assume that $0 < t_0 \le x \le t \le T$ and $u \in C_+(J)$. Applying (h1) and noticing that $f(\cdot, u)$ is nonnegative valued, we get

$$\int_0^T h(s)\, ds \ge \int_0^t f(s,u)\, ds \ge \int_0^x f(s,u)\, ds \ge \int_0^{t_0} f(s,0)\, ds > 0.$$

This result implies by (27.3) and (h2) that

$$0 < G0(t_0) \le Gu(x) \le Gu(t) \le b. \tag{27.5}$$

Because $\frac{1}{q} \in L^\infty_{loc}(0,\infty)$ by (h2), there is a positive constant $M(t_0)$ such that

$$\frac{1}{q(y)} \le M(t_0) \quad \text{for a.e.} \quad y \in [G0(t_0), b]. \tag{27.6}$$

It then follows from (27.5) and (27.6) that

$$\frac{1}{M(t_0)} \le q(y) \quad \text{for a.e. } y \in [Gu(x), Gu(t)].$$

Applying this result and (27.3), we then have

$$\frac{Gu(t) - Gu(x)}{M(t_0)} = \int_{Gu(x)}^{Gu(t)} \frac{1}{M(t_0)}\, dy$$

$$\leq \int_{Gu(x)}^{Gu(t)} q(y)\, dy = \int_x^t f(s,u)\, ds \leq \int_x^t h(s)\, ds.$$

This result and (27.5) imply that (27.4) holds.

Denote by $AC_{loc}^{+}(0,T]$ the set of all $u \in C_+(J)$ which are locally absolutely continuous on $(0,T]$.

Lemma 2. *Assume that the hypotheses (h0)–(h2) hold. Then $u \in AC_+(0,T]$ is a solution of the Cauchy problem (27.1) if and only if u is a fixed point of the operator $G : C_+(J) \to C_+(J)$ defined by (27.3).*

Proof. Assume first that $u \in AC_{loc}^{+}(0,T]$ is a solution of (27.1). The hypotheses (h1) and (h2) and the differential equation of (27.1) imply that $u'(t) \geq 0$ a.e. in J. Thus, u is in $C_+(J)$, and satisfies by (27.1) the integral equation

$$\int_0^t q(u(s))u'(s)\, ds = \int_0^t f(s,u)\, ds, \quad t \in J. \tag{27.7}$$

Because $q \in L_{loc}^1(\mathbb{R}_+)$, $u \in AC_{loc}^{+}(0,T]$, and u is monotone, we can change by ([McSh74], 38, 3-4) the variable on the left-hand side of (27.7) to obtain

$$\int_{u(t_0)}^{u(t)} q(y)\, dy = \int_{t_0}^t f(s,u)\, ds, \quad 0 < t_0 \leq t \leq T.$$

Noticing that $u(0) = 0$, we obtain

$$\int_0^{u(t)} q(y)\, dy = \int_0^t f(s,u)\, ds, \quad t \in J. \tag{27.8}$$

It then follows from (27.3) and (27.8) that $u = Gu$, i.e., u is a fixed point of G. Conversely, assume that u is a fixed point of G, defined by (27.3). Since $u = Gu$, it follows from (27.4) that

$$0 \leq u(t) - u(x) \leq M(t_0) \int_x^t h(s)\, ds \ \text{ whenever } 0 < t_0 \leq x \leq t \leq T. \tag{27.9}$$

Since the function $t \mapsto \int_0^t h(s)\, ds$ is absolutely continuous on J, it follows from (27.9) that u is absolutely continuous on $[t_0,T]$, for each $t_0 \in (0,T)$. Moreover, u is increasing by (27.9), so that we can change by ([McSh74], 38, 3-4) the variable on the left-hand side of

$$\int_{u(t_0)}^{u(t)} q(y)\ dy = \int_{t_0}^{t} f(s,u)\ ds, \quad 0 < t_0 \le x \le t \le T, \tag{27.10}$$

and obtain

$$\int_{t_0}^{t} q(u(s))u'(s)\ ds = \int_{t_0}^{t} f(s,u)\ ds, \quad t \in J. \tag{27.11}$$

Since (27.11) holds for any $t_0 \in (0,T)$, differentiating sidewise with respect to t, we see that the differential equation of (27.1) holds for a.e. $t \in J$. Moreover, it follows from (27.8) as $t = 0$ that $u(0) = 0$. Thus, $u \in AC^+_{loc}(0,T]$ is a solution of the Cauchy problem (27.1).

The following fixed point result is a consequence of Theorem A.2.1. of [CaHe00].

Lemma 3. *Assume that $G : C_+(J) \to C_+(J)$ is increasing, i.e., $Gu \le Gv$ whenever $u \le v$, that the range $G[C_+(J)]$ of G is order bounded, and that each well-ordered chain of $G[C_+(J)]$ has a supremum in $C_+(J)$, and each inversely well-ordered chain has an infimum in $C_+(J)$. Then G has least and greatest fixed points, and they are increasing with respect to G.*

Now we are ready to prove our main existence result for the Cauchy problem (27.1).

Theorem 1. *Assume that the hypotheses (h0)–(h2) hold. Then the Cauchy problem (27.1) has least and greatest solutions in $AC^+_{loc}(0,T]$, and they are increasing with respect to f.*

Proof. It suffices to show that the operator G, defined by (27.3), satisfies the hypotheses of Lemma 3. If $u \le v$ in $C_+(J)$, it follows from (27.3) by the hypothesis (h1) that

$$\int_0^{Gu(t)} q(y)\ dy = \int_0^t f(s,u)\ ds \le \int_0^t f(s,v)\ ds = \int_0^{Gv(t)} q(y)\ dy, \quad t \in J.$$

This implies that $Gu(t) \le Gv(t)$ for each $t \in J$, whence G is increasing. Since (27.5) holds for each $u \in C_+(J)$, then the range of G is order bounded. It follows from (27.4) that for each $t_0 \in (0,T)$ the restrictions of Gu, $u \in C_+(J)$, to $[t_0,T]$ form an equicontinuous set. Moreover, (27.3) implies by (h0) that

$$\int_0^{Gu(t)} q(y)\ dy \le \int_0^t h(s)\ ds, \quad u \in C_+(J),\ t \in J.$$

Thus the functions Gu, $u \in C_+(J)$, are equicontinuous at 0. Consequently, $G[C_+(J)]$ is an equicontinuous subset of $C_+(J)$. It then follows from Proposition 1.3.8 of [HeiLa94] and its dual that each well-ordered chain of $G[C_+(J)]$ has a supremum in $C(J)$ and each inversely well-ordered chain of $G[C_+(J)]$

has an infimum in $C(J)$. Because each Gu is nonnegative valued, then these supremums and infimums belong to $C_+(J)$.

The preceding proof shows that the operator G defined by (27.3) satisfies the hypotheses of Lemma 3, whence G has a least fixed point u_* and a greatest fixed point u^*. According to Lemma 2, u_* and u^* are least and greatest absolutely continuous solutions of the Cauchy problem (27.1).

The last assertion is an easy consequence of the last conclusion of Lemma 3 and the definition of G.

Example 1. Determine the least and greatest solutions to the Cauchy problem

$$u'(t) = \frac{3}{2}(1+u(t)^{-2})\,\cos(t)\,\overline{\arctan}\left(3D(t) + [\int_0^1 u(t)\,dt]\right),\ u(0)=0, \quad (27.12)$$

for a.e. $t \in J = [0,1]$, where $[\cdot]$ denotes the greatest integer function and D is the Dirichlet function.

Solution. Problem (27.12) can be rewritten in the form (27.1), where

$$q(y) = \frac{y^2}{1+y^2},\ f(t,u) = \frac{3}{2}\cos(t)\,\overline{\arctan}\left(3D(t) + [\int_0^1 u(t)\,dt]\right). \quad (27.13)$$

Simple calculations show that (27.13) defines mappings $q : \mathbb{R}_+ \to \mathbb{R}_+$ and $f : J \times C_+(J) \to \mathbb{R}$ which satisfy the hypotheses (h0)–(h2). It then follows from Theorem 1 that the Cauchy problem (27.1) has least and greatest absolutely continuous solutions. By Lemma 2 the solutions of (27.12) are the same as the fixed points of the operator $G : C_+(J) \to C_+(J)$ given by (27.3) with q and f defined by (27.13), or equivalently,

$$Gu(t) = \overline{\arctan}\,u(t) + \frac{3}{2}\int_0^t \left(\cos(s)\,\overline{\arctan}(3D(s) + [\int_0^1 u(t)\,dt])\right)\,ds. \quad (27.14)$$

By the proof of Lemma 3 (Theorem A.2.1. of [CaHe00]) the least solution u_* is the maximum of the well-ordered chain C in $C_+(J)$ which satisfies

$$a = \min C,\ \text{and}\ a < u \in C \ \ <=> \ \ u = \sup G[\{v \in C \mid v < u\}], \quad (27.15)$$

where $a \equiv 0$. It is easy to show that the least elements of C are the successive approximations: $u_{n+1} = Gu_n$, $n \in \mathbb{N}$, $u_0 = a$. Calculating these approximations it turns out that for $n \geq 2$ they satisfy

$$u_{n+1}(t) = \overline{\arctan}\,u_n(t) + \frac{3}{2}\,\overline{\arctan}(4)\,\sin(t), \quad t \in J.$$

Because the sequence $(u_n)_{n=1}^{\infty}$ is increasing and belongs to $G[C_+(J)]$, which is an equicontinuous set by the proof of Theorem 1, it converges uniformly on J to the solution u_ω of the equation

$$u(t) = \overline{\arctan}\, u(t) + \frac{3}{2}\,\overline{\arctan}(4)\,\sin(t), \quad t \in J. \tag{27.16}$$

Since $\int_0^1 u_\omega(t)\,dt \approx 1.97$, then $[\int_0^1 u_\omega(t)\,dt] = 1$. Consequently, if f is defined by (27.13), then

$$f(t, u_\omega) = \frac{3}{2}\,\overline{\arctan}(4)\,\cos(t), \text{ a.e. in } J.$$

Thus,

$$\int_0^t f(s, u_\omega)\,ds = \frac{3}{2}\,\overline{\arctan}(4)\,\sin(t), \quad t \in J,$$

whence the solution u_ω of (27.16) is a fixed point of G, i.e., a solution of

$$u(t) = \overline{\arctan}\, u(t) + \frac{3}{2}\int_0^t \left(\cos(s)\overline{\arctan}(3D(s) + [\int_0^1 u(t)\,dt])\right) ds \tag{27.17}$$

on J. Moreover, the above reasoning shows that $u_\omega = \max C$, so that $u_\omega = u_*$. In particular, (27.16) is the implicit representation of the least absolutely continuous solution of the Cauchy problem (27.12).

Similarly, the greatest solution u^* of (27.12) is the minimum of the inversely well-ordered chain D in Y which satisfies

$$b = \max D, \text{ and } b > u \in D \iff u = \inf G[\{v \in C \mid u < v\}], \tag{27.18}$$

where G is defined by (27.14) and $b \equiv 3$. The greatest elements of D are the successive approximations: $v_{n+1} = Gv_n$, $n \in \mathbb{N}$, $v_0 = b$. Calculating these approximations, it turns out that for $n \geq 2$ they satisfy

$$v_{n+1}(t) = \overline{\arctan}\, v_n(t) + \frac{3}{2}\,\overline{\arctan}(5)\,\sin(t), \quad t \in J.$$

The sequence $(v_n)_{n=0}^\infty$ is decreasing and equicontinuous, whence it converges uniformly on J to the solution v_ω of the equation

$$u(t) = \overline{\arctan}\, u(t) + \frac{3}{2}\,\overline{\arctan}(5)\,\sin(t), \quad t \in J. \tag{27.19}$$

Since $\int_0^1 v_\omega(t)\,dt \approx 2.01$, then $[\int_0^1 v_\omega(t)\,dt] = 2$. Consequently, if f is defined by (27.13), then

$$f(t, v_\omega) = \frac{3}{2}\,\overline{\arctan}(5)\,\cos(t), \text{ a.e. in } J.$$

Thus,

$$\int_0^t f(s, v_\omega)\,ds = \frac{3}{2}\,\overline{\arctan}(5)\,\sin(t), \quad t \in J,$$

whence the solution $u = v_\omega$ of (27.19) is a fixed point of G, and hence a solution of (27.17) and (27.12). Moreover, $v_\omega = \min D$, so that $v_\omega = u^*$. Thus, (27.19) is the implicit representation of the greatest absolutely continuous solution of (27.12).

Remark 1. The calculations needed in (1) are carried out by using Maple 9 and simple Maple programs. The solutions u_* and u^* are shown in Figure 27.1.

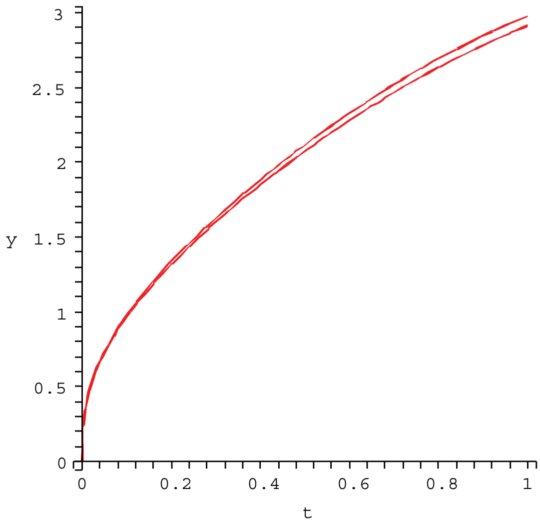

Fig. 27.1. Least and greatest solutions of (27.12).

27.3 Cauchy–Nicoletti Problem

Next we will study the Cauchy–Nicoletti problem

$$q_i(u_i(t))u_i'(t) = f_i(t,u) \text{ for a.e. } t \in J = [0,T],\ u_i(t_i) = c_i,\ i = 1,...,n, \tag{27.20}$$

where $0 = t_1 < t_2 < \cdots < t_n = T$, and $c = (c_1,...,c_n) \in \mathbb{R}^n$. For other studies of the Cauchy–Nicoletti problem see, e.g., [BlWa76], [Ka04], and [Sei82].

Denote $C_n(J) = \{u = (u_1,...,u_n) : J \to \mathbb{R}^n \mid u_i \text{ is continuous}, i = 1,...,n\}$, and equip $C_n(J)$ with the partial ordering defined by

$$u \le v \iff u_i \le v_i,\ i = 1,...,n.$$

The functions $q_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $f_i : J \times C_n(J) \to \mathbb{R}_+,\ i = 1,...,n$, are assumed to satisfy the following:

(H0) $f_i(\cdot,u)$ is Lebesgue measurable for all $u \in C_n(J)$, $q_i \in L^1_{loc}(c_i,\infty) \cap L^1_{loc}(-\infty,c_i)$, and $\frac{1}{q_i} \in L^\infty_{loc}(c_i,\infty) \cap L^\infty_{loc}(-\infty,c_i)$ for $i=2,...,n-1$, $q_1 \in L^1_{loc}(c_1,\infty)$, $q_n \in L^1_{loc}(-\infty,c_n)$, $\frac{1}{q_1} \in L^\infty_{loc}(c_1,\infty)$, $\frac{1}{q_n} \in L^\infty_{loc}(-\infty,c_n)$;

(H1) there exist $h_i \in L^1(J)$, $d_i < c_i$, and $b_i > c_i$ such that $f_i(\cdot,u) \le h_i$ for $u \in C_n(J)$, $(d_1,...,d_n) = d \le u \le b = (b_1,...,b_n)$, and $\int_0^{t_i} h_i(s)\, ds \le \int_{d_i}^{c_i} q_i(y)\, dy$ for $i=2,...,n$, $\int_{t_i}^T h_i(s)\, ds \le \int_{c_i}^{b_i} q_i(y)\, dy$ for $i=1,...,n-1$;

(H2) $\int_{t_i}^t f_i(s,u)\, ds \le \int_{t_i}^t f_i(s,v)\, ds$ and $|\int_{t_i}^t f_i(s,u)\, ds| > 0$ for $t \in J$, $t \ne t_i$ and for $u,v \in C_n(J), d \le u \le v \le b$.

We note that the first inequality in (H2) holds for a 2-point problem, $t_1 = 0, t_2 = T$, if, for example, $f_1(t,u)$ is increasing and $f_2(t,u)$ decreasing in u for $t \in J$. In a 3-point problem we might have $f_1(t,u)$ increasing in u for $t \in J$, $f_2(t,u)$ increasing in u for $t \in [0,t_2]$ and decreasing for $t \in [t_2,T]$ and $f_3(t,u)$ decreasing in u for $t \in J$.

By a solution of problem (27.20) we mean a function $u \in C_n(J)$ such that every component function u_i is locally absolutely continuous on $(t_i,T]$, $i = 1,...,n-1$, and on $[0,t_i)$, $i = 2,...,n$, and u satisfies (27.20). Denote $[d,b] = \{u \in C_n(J) | d \le u \le b\}$.

Lemma 4. *Let the hypotheses (H0)–(H2) hold. Then the equations*

$$\int_{c_i}^{G_i u(t)} q_i(y)\, dy = \int_{t_i}^t f_i(s,u)\, ds, \quad t \in J,\ u \in [d,b],\ i = 1,...,n, \qquad (27.21)$$

define an increasing mapping $G : [d,b] \to [d,b]$, $G = (G_1,...,G_n)$. Moreover, problem (27.20) has a solution $u \in [d,b]$ if and only if u is a fixed point of G.

Proof. Let $u \in C_n(J)$, $d \le u \le b$. Using assumptions (H0) and (H1), it can be proved, similarly as in the proof of Lemma 2, that $G_i u$ is defined on J, $i = 1,2,...,n$. Moreover, we may choose $\bar{t}_{i0}$ and $\underline{t}_{i0}$ such that for $t_i < \bar{t}_{i0} \le x \le t \le T$, $i = 1,...,n-1$, we have

$$\begin{aligned}\int_{c_i}^{b_i} q_i(y)\, dy \quad &\ge \quad \int_{t_i}^T h_i(s)\, ds \ge \int_{t_i}^t f_i(s,u)\, ds \\ &\ge \quad \int_{t_i}^x f_i(s,u)\, ds \ge \int_{t_i}^{\bar{t}_{i0}} f_i(s,d)\, ds > 0,\end{aligned}$$

which implies that

$$c_i < G_i d(\bar{t}_{i0}) \le G_i u(x) \le G_i u(t) \le b_i$$

and for $0 \le t \le x \le \underline{t}_{i0} < t_i$, $i = 2,...,n$, we have

$$\begin{aligned}\int_{c_i}^{d_i} q_i(y)\, dy \quad &\le \quad \int_{t_i}^0 h_i(s)\, ds \le \int_{t_i}^t f_i(s,u)\, ds \\ &\le \quad \int_{t_i}^x f_i(s,u)\, ds \le \int_{t_i}^{\underline{t}_{i0}} f_i(s,b)\, ds < 0,\end{aligned}$$

which implies that

$$d_i \le G_i u(t) \le G_i u(x) \le G_i b(\underline{t}_{i0}) \le c_i.$$

Hence, $d \le Gu \le b$. For $d \le u \le v \le b$ we have

$$\begin{aligned} \int_{c_i}^{G_i u(t)} q_i(y)\ dy &= \int_{t_i}^{t} f_i(s,u)\ ds \le \int_{t_i}^{t} f_i(s,v)\ ds \\ &= \int_{c_i}^{G_i v(t)} q_i(y)\ dy \quad t \in J,\ i = 1,...,n, \end{aligned}$$

which implies that $G_i u(t) \le G_i v(t), i = 1,...,n$, i.e., that $G : [d,b] \to [d,b]$ is increasing.

As in the proof of Lemma 2, it can be proved that the functions $G_i u$ are absolutely continuous on closed subintervals of $(t_i, T],\ i = 1,...,n-1$, and $[0,t_i),\ i = 2,...,n$, and again using the change of variables that problem (27.20) has a solution $u \in [d,b]$ if and only if u is a fixed point of G.

Theorem 2. *Assume that the hypotheses (H0)–(H2) hold. Then the Cauchy–Nicoletti problem (27.20) has least and greatest solutions in the segment* $[d,b]$ *of* $C_n(J)$.

Proof. The result is a consequence of Lemma 4, Theorem A.2.1 of [CaHe00] and of Proposition 1.3.8 of [HeiLa94] when we note that in Lemma 3 $G[C_+(J)]$ can be replaced by $[d,b]$, and similarly as in the proof of Theorem 1 it can be shown that $G[d,b]$ is an equicontinuous subset of $C_n(J)$.

As an example of a Cauchy–Nicoletti problem, we will consider a singular 2-point boundary value problem:

$$q(u'(t))u''(t) = f(t,u) \text{ for a.e. } t \in J = [0,1],\ u(0) = u_0,\ u'(1) = u_1, \quad (27.22)$$

where $u_0 \in \mathbb{R},\ u_1 > 0$, and $q : \mathbb{R}_+ \to \mathbb{R}_+$ and $f : J \times C_2(J) \to \mathbb{R}_+$ satisfy
(f0) $f(\cdot,u)$ is Lebesgue measurable and $f(\cdot,u) \le h \in L^1(J)$ for all $u \in C_2(J)$;
(f1) $0 < \int_t^1 f(s,u)\ ds \le \int_t^1 f(s,v)\ ds$ whenever $u \le v$ in $C_2(J)$, $t \in [0,1)$;
(q1) $q \in L^1_{loc}(-\infty,u_1)$, $\frac{1}{q} \in L^\infty_{loc}(-\infty,u_1)$, and $\int_a^{u_1} q(y)\ dy \ge \int_0^1 h(s)\ ds$ for some $a \in (0,u_1)$.

Corollary 1. *Assume that the hypotheses (f0), (f1), and (q1) hold. Then there exist such* $d = (d_1,d_2)$ *and* $b = (b_1,b_2) \in \mathbb{R}_2$ *that the boundary value problem (27.22) has least and greatest solutions satisfying* $d \le (u,u') \le b$.

Proof. By choosing $n = 2$, $t_1 = 0$, $t_2 = 1$, $u_1 = u$, $u_2 = u'$, $f_1(t,u) = u_2(t)$, $f_2(t,u) = f(t,u)$, $c_1 = u_0$, $c_2 = u_1$, $q_1 \equiv 1$, and $q_2 = q$, the problem (27.22) is converted to problem (27.20). Now choose b_1 and b_2 such that $u_0 + b_2 \leq b_1$ and let $d_1 < b_1, d_2 = a$, and $h_2 = h$. Then for $(u, u') \in [d, b] = [(d_1, d_2), (b_1, b_2)]$ we have $f_1(t, u) \leq h_1(t)$ for $h_1(t) \equiv b_2$ and $\int_0^1 h_1(s)\ ds \leq \int_{c_1}^{b_1} q_1(y)\ dy$ is equivalent to $u_0 + b_2 \leq b_1$. Since $\int_0^t f_1(s, u)\ ds = \int_0^t u_2(s)\ ds > 0,\ t \in (0, 1]$, and (f1) holds, we note that the assumptions (H0)–(H2) are satisfied, and the conclusion follows from Theorem 2.

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28

Asymptotic Behavior of the Solution of an Elliptic Pseudo-Differential Equation Near a Cone

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28.1 Preliminaries

We consider the equation

$$(Au_+)(x) = f(x), \quad x \in C_+^a, \tag{28.1}$$

where A is a pseudo-differential operator with symbol $A(\xi)$ satisfying the condition

$$c_1 \leq \left|A(\xi)(1+|\xi|)^{-\alpha}\right| \leq c_2, \quad \forall \xi \in \mathbb{R}^m,$$

and C_+^a is the cone $\{x \in \mathbb{R}^m : \ x_m > a|x'|, \ x' = (x_1, \dots, x_{m-1}), \, a > 0\}$.

Definition 1. *The symbol $A(\xi)$ admits a wave factorization with respect to the cone C_+^a if it can be represented in the form*

$$A(\xi) = A_{\neq}(\xi) A_{=}(\xi),$$

where the factor $A_{\neq}(\xi)$ has the following properties:

1) $A_{\neq}(\xi)$ is defined on $\mathbb{R}^m$ except possibly at the points $\{x \in \mathbb{R}^m : \ a^2 x_m^2 = |x'|^2\}$;

2) $A_{\neq}(\xi)$ admits an analytical continuation into the radial tube domain $T(\overset{*}{C}{}_+^a)$ [V64] over the cone $\overset{*}{C}{}_+^a = \{x \in \mathbb{R}^m : \ ax_m > |x'|\}$, satisfying the estimate

$$\left|A_{\neq}^{\pm}(\xi + i\tau)\right| \leq c(1 + |\xi| + |\tau|)^{\pm\kappa}, \quad \forall \tau \in \overset{*}{C}{}_+^a.$$

Analogous properties must have the factor $A_{=}(\xi)$ with $\alpha - \kappa$ instead of κ and $\overset{*}{C}{}_-^a = -\overset{*}{C}{}_+^a$ instead of $\overset{*}{C}{}_+^a$.

The number κ is called the index of the wave factorization.

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28.2 Solvability

Let $H^s(\mathbb{R}^m)$ be the vector space of functions with norm

$$||u||_s^2 = \int\limits_{\mathbb{R}^m} |\tilde{u}(\xi)|^2(1+|\xi|)^{2s}d\xi,$$

where "$\sim$" denotes the distributional Fourier transform, and $H^s(C_+^a)$ is the subspace of $H^s(\mathbb{R}^m)$ of all elements with support in $\overline{C_+^a}$.

We define an integral operator G_m (at first for functions from the Schwartz class $S(\mathbb{R}^m)$) by the formula

$$(G_m u)(x) = \lim_{\tau\to 0+} \int\limits_{\mathbb{R}^m} \frac{u(y', y_m)dy}{[(x'-y')^2 - a^2(x_m - y_m + i\tau)^2]^{m/2}},$$

which can be extended to a bounded operator $L_2(\mathbb{R}^m) \to L_2(\mathbb{R}^m)$. Such an operator will help us construct the solution of equation (28.1). The right-hand side of (28.1) is assumed to belong to the space $\overset{\circ}{H}{}^{s-\alpha}(C_+^a)$ consisting of functions $f \in H^{s-\alpha}(C_+^a)$ admitting a continuation $\ell f \in H^{s-\alpha}(\mathbb{R}^m)$, with the norm

$$||f||_s^+ = \inf ||\ell f||_s,$$

and the infimum is taken for all continuations ℓ.

Theorem 1. *If the symbol $A(\xi)$ admits a wave factorization with respect to the cone C_+^a with index $\kappa = 0$, then equation (28.1) with arbitrary right-hand side $f \in \overset{\circ}{H}{}^{s-\alpha}(C_+^a)$ has a unique solution $u_+ \in H^s(C_+^a)$, which can be written in the form*

$$\tilde{u}_+(\xi) = A_{\neq}^{-1}(\xi) G_m A_{=}^{-1}\widetilde{\ell f}, \tag{28.2}$$

and satisfies the a priori estimate

$$||u||_s \le c||f||_{s-\alpha}^+.$$

28.3 Asymptotics

First, we consider $m = 2$. The operator G_2 defined by ($\tau > 0$ is fixed)

$$(G_2 u)(x) = \lim_{\tau\to 0+} \int\limits_{\mathbb{R}^2} \frac{u(y_1, y_2)\,dy_1 dy_2}{(x_1 - y_1)^2 - a^2(x_2 - y_2 + i\tau)^2},$$

after the change of variables

$$(x_1 - y_1)^2 - a^2(x_2 - y_2 + i\tau)^2$$
$$= (x_1 - y_1 - a(x_2 - y_2 + i\tau))\,(x_1 - y_1 + a(x_2 - y_2 + i\tau))$$
$$= (z_1 - \eta_1)(z_2 - \eta_2),$$

where $z_1 = x_1 - ax_2 - ai\tau$, $z_2 = x_1 + ax_2 + ai\tau$, $\eta_1 = y_1 - ay_2$, $\eta_2 = y_1 + ay_2$, will take the form

$$(\tilde{G}_2 u)(\xi_1, \xi_2) = \lim_{\tau \to 0+} \frac{1}{2a} \int\limits_{\mathbb{R}^2} \frac{u(\eta_1, \eta_2)\, d\eta_1 \eta_2}{(z_1 - \eta_1)(z_2 - \eta_2)}, \tag{28.3}$$

where $\xi_1 = x_1 - ax_2$ and $\xi_2 = x_1 + ax_2$.

We remark that such a linear transformation maps the first quadrant of the plane onto the second one.

As in [G77], we can introduce a piecewise analytical function

$$\Phi(z_1, z_2) = \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} \frac{u(\eta_1, \eta_2)\, d\eta_1 d\eta_2}{(z_1 - \eta_1)(z_2 - \eta_2)}$$

for a suitable function $u(\eta_1, \eta_2)$, and then in formula (28.3) we have the boundary values Φ^{-+}, which consist of four summands (up to constants):

$$\Phi^{-+}(\xi_1, \xi_2) = -u(\xi_1, \xi_2) + \frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{u(\eta_1, \eta_2)\, d\eta_1}{\xi_1 - \eta_1}$$
$$- \frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{u(\xi_1, \eta_2)\, d\eta_2}{\xi_2 - \eta_2} - \frac{1}{\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{u(\eta_1, \eta_2)\, d\eta_1 d\eta_2}{(\xi_1 - \eta_1)(\xi_2 - \eta_2)}. \tag{28.4}$$

This is the basic formula that will help us obtain an asymptotic expansion of the solution near the boundary.

The second and third summands are Cauchy-type integrals (Hilbert transforms), and for such functions specific methods are already developed (see [E81]). The first part of (28.4) is a smooth function. The last integral is a combination of the second and third integrals, and we can apply to it the same approach.

We consider a pseudo-differential operator with symbol $A_{\neq}^{-1}(\xi)$. Its homogeneity order is equal to $-\kappa$. Roughly speaking, this means that the corresponding integral operator looks like the convolution operator

$$F^{-1}_{\xi \to y}\left[A_{\neq}^{-1} \tilde{u}\right] = \int\limits_{\mathbb{R}^2} K(x - y) u(y)\, dy,$$

assuming that the integral exists (at least in the Calderon–Zygmund sense), and its kernel satisfies the estimate $|K(x)| \le c|x|^{-m-\kappa}$.

Suppose now that the function $u(y)$ is sufficiently smooth and has compact support in C_+^a. Then

$$v(x) = \int\limits_{\mathbb{R}^2} K(x-y)u(y)\,dy = \int\limits_{C_+^a} K(x-y)u(y)\,dy,$$

and if $x \notin C_+^a$, then $v(x) = 0$, i.e., $\operatorname{supp} v(x) \equiv D \subset C_+^a$.

Let $r(x)$ be the distance from x to ∂C_+^a, and suppose that $r(x)$ is so small that $B\left(x, \frac{r(x)}{2}\right) \subset D$; then

$$|v(x)| \leq c \int\limits_{D \setminus B\left(x, \frac{r(x)}{2}\right)} \frac{|u(y)|\,dy}{|x-y|^{m+\kappa}}.$$

If $y \in D \setminus B\left(x, \frac{r(x)}{2}\right)$, then

$$|x-y| \leq |x-y| + \frac{r(x)}{2} \leq 2|x-y|,$$

so that

$$|v(x)| \leq c(u) \int\limits_{D \setminus B\left(x, \frac{r(x)}{2}\right)} \frac{dy}{\left(|x-y| + \frac{r(x)}{2}\right)^{m+\kappa}},$$

where $c(u)$ is a constant depending on u.

In the last integral, using spherical coordinates we can easily obtain the estimate

$$|v(x)| \leq c(u) \int\limits_{\frac{r(x)}{2}}^{d} \frac{dt}{\left(t + \frac{r(x)}{2}\right)^{m+\kappa}},$$

from which it immediately follows that

$$|v(x)| \leq c(u) \begin{cases} r(x), & \kappa > -1, \\ \ln r(x), & \kappa = -1, \\ 1, & \kappa < -1. \end{cases} \tag{28.5}$$

As mentioned in [E81], the solution $u_+(x)$ under $\kappa \leq -\frac{1}{2}$ will, in general, be a distribution, which needs further consideration.

Since the operator G_m acts like a multiplier in Fourier images and does not change the smoothness properties of functions, it follows that for a sufficiently smooth right-hand side f, the solution $u_+(x)$ of equation (28.1) will be smooth everywhere except perhaps at boundary points with growth (28.5).

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29

Averaging Normal Forms for Partial Differential Equations with Applications to Perturbed Wave Equations

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29.1 Introduction

Normalization and normal forms play an important part in mathematical analysis and algebra. For instance, $n \times n$-matrices can be put in Jordan normal form. Such an example also makes it clear that normalization is not a unique procedure as the choice of normalization of matrices depends on its purpose. In the case of matrices there is a vast literature with many possibilities, but in all special cases and in other mathematical problems as well, the general aim of normalization is a simplification of the object by transformation.

In the case of ordinary differential equations (ODEs) of the form

$$\dot{x} = \varepsilon f(t, x),$$

with ε a small positive parameter, averaging normalization can be summarized as follows. Assume that the limit

$$f^0(z) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(z, s) ds$$

exists. Introduce the averaging normalization transformation

$$x(t) = z(t) + \varepsilon \int_0^t (f(z, s) - f^0(z)) ds.$$

With a few assumptions and using elementary calculations, one finds for z the equation

$$\dot{z} = \varepsilon f^0(z) + \varepsilon^2 f^1(t, z, \varepsilon).$$

The equation has been normalized to $O(\varepsilon)$; the simplification is the removal of the variable t and what are called nonresonant terms from the equation to $O(\varepsilon)$. With additional assumptions one can extend the normalization to $O(\varepsilon^2)$ and higher order.

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This procedure for ODEs is well known; for a description and references see [SaVeMu07]. The aim of this chapter is to describe in a tutorial way the normalization procedure for a number of partial differential equations (PDEs) (Sections 29.2–29.4) and to discuss a few new examples. Averaging normalization for PDEs is of more recent date, and the theory is far from complete. Additional material on this topic can be found in [Ve05].

29.2 Normal Forms for Parabolic Equations

A typical problem formulation is to consider an equation of the form

$$u_t + Lu = \varepsilon f(u), \ t \geq 0, \tag{29.1}$$

with given initial and boundary values, L a linear operator, u an element of a suitable function space, and $f(u)$ representing the linear and nonlinear perturbation terms.

The first step is to solve the "unperturbed" problem

$$\frac{\partial u_0}{\partial t} + Lu_0 = 0, \ t \geq 0, \tag{29.2}$$

with the given initial and boundary values. If the domain has a simple geometrical shape like a circle or a rectangle, this may not present difficulties. In real-life problems, the domain is more complicated, and one has to resort to numerical methods.

One may well ask: if we have to use numerical methods for the unperturbed problem, why would I not use these methods directly for the perturbed problem? The answer is that in evolution equations, long-time numerical integrations may present a big obstacle. Averaging weeds out the short-periodic or short-oscillatory terms, and this improves the interval of validity of the computations enormously. So, even if we have to perform numerical integration of the unperturbed and the normalized equation(s), this may still be an effective procedure.

29.2.1 Advection

To focus the discussion, we consider a problem from [Kr91]. In this case, the domain is two dimensional, and the unperturbed equation is

$$\frac{\partial C_0}{\partial t} + \nabla(v_0.C_0) = 0, \ t \geq 0. \tag{29.3}$$

The equation describes advection for transport problems. We will consider the application to tidal basins like the North Sea. In this case, the two-dimensional vector $v_0 = v_0(x, y, t)$ is the basic periodic flow due to tidal currents that is supposed to be known. The transportation of material, e.g., sediment or chemicals, is represented by the concentration C_0; the term $\nabla(v_0.C_0)$ represents the advection with the flow.

In the application to tidal basins, one often considers the basic flow to be divergence free, so

$$\nabla.v_0 = 0.$$

The unperturbed equation becomes

$$\frac{\partial C_0}{\partial t} + v_0.\nabla C_0 = 0,\ t \geq 0. \tag{29.4}$$

Equation (29.4) is a first order equation which can be integrated along the characteristics $P(t)(x, y)$, in this case also called streamlines. Due to the uniqueness of the solutions of equation (29.4), $P(t)(x, y)$ is an invertible map with inverse $Q(t)(x, y)$.

The solution C_0 is constant along the characteristics, so on adding the initial condition

$$C_0(x, y, 0) = \gamma(x, y),$$

we find the solution

$$C_0(P(t)(x, y), t) = \gamma(x, y),$$

so that

$$C_0(x, y, t) = \gamma(Q(t)(x, y)). \tag{29.5}$$

29.2.2 Advection–Diffusion

Several types of perturbations of advection are possible. For the application in [Kr91], one considers the fact that tidal basins are open. This results in a small rest stream so that the tidal current is perturbed:

$$v(x, y, t) = v_0(x, y, t) + \varepsilon v_1(x, y).$$

The rest stream is assumed to be divergence free: $\nabla.v_1 = 0$.

A second perturbation arises from diffusion in the basin, expressed by the term $\varepsilon \Delta C$. The equation to be studied is then

$$\frac{\partial C}{\partial t} + v_0.\nabla C + \varepsilon v_1.\nabla C = \varepsilon \Delta C,\ t \geq 0, \tag{29.6}$$

with given initial condition $C(x, y, 0) = \gamma(x, y)$. This is still a linear problem. Note that the tidal current has a period of nearly 12 hours, and the effect of small diffusion entails a timescale of 6-12 months.

29.2.3 The Standard Form for Averaging

Using variation of constants, we obtain a slowly varying system. The transformation is

$$C(x, y, t) = F(Q(t)(x, y), t).$$

If $\varepsilon = 0$, we have $C = C_0, F = \gamma$, and C_0 is constant on the characteristics. If $\varepsilon > 0$ and small, this results in a slowly varying F. By differentiation we obtain an equation of the form

$$\frac{\partial F}{\partial t} = \varepsilon L(t)F$$

with initial condition $F(x, y, 0) = \gamma(x, y)$. The linear operator $L(t)$ is computed using the perturbation terms and the unperturbed solution (from P and Q). In this problem $L(t)$ is uniformly elliptic and T-periodic in t. Averaging over t produces the approximating system

$$\frac{\partial \bar{F}}{\partial t} = \varepsilon L^0 \bar{F}$$

with initial value $\bar{F}(x, y, 0) = \gamma(x, y)$ and

$$L^0 = \frac{1}{T}\int_0^T L(t)dt.$$

In [Kr91] it is proved that $||F - \bar{F}||_\infty = O(\varepsilon)$ on the long timescale $1/\varepsilon$. For the corresponding approximation $\bar{C}$ of C, we have the same estimate. In [Kr91] a number of extensions of the theory are also indicated.

29.2.4 Reactions and Sources

An extension with interesting aspects is to consider reactions of chemicals or sediment using a reaction term $f(C)$. It is also natural to include localized sources indicated by $B(x, y, t)$ which, in the case of tidal basins, can be interpreted as periodic dumping of chemicals or sediment in the basin. Following [HeKrVe95] the equation becomes

$$\frac{\partial C}{\partial t} + v_0.\nabla C + \varepsilon v_1.\nabla C = \varepsilon \Delta C + \varepsilon f(C) + \varepsilon B(x, y, t), \; t \geq 0, \tag{29.7}$$

with given initial condition $C(x, y, 0) = \gamma(x, y)$. The reaction term will in general be nonlinear, for instance, $f(C) = aC^2$ or $f(C) = aC^5$, depending on the type of reaction. $B(x, y, t)$ is periodic in t. Using again variation of constants, we obtain from equation (29.7) a perturbation equation in the same way as shown above, but with a more complicated operator $L(t)$.

As the tidal period of $v_0(x, y, t)$ is near to 12 hours, it is natural to assume a common period T with the dumping process indicated by $B(x, y, t)$.

Averaging produces an approximation $\bar{C}$ of the solution C of the initial value problem for equation (29.7). Interestingly, the result is stronger than in the case without the source term. One can prove that $\bar{C}$ converges to the solution $\bar{C}_0$ of a time-independent boundary value problem, while C converges to a T-periodic solution which is ε-close to $\bar{C}_0$ *for all time*. The proof is based on a maximum principle and the use of suitable subsolutions and supersolutions of equation (29.7). For details, see [HeKrVe95].

29.3 Two Basic Normal Form Theorems

Consider the semilinear initial value problem

$$\frac{dw}{dt} + \mathcal{A}w = \varepsilon f(w, t, \varepsilon), \quad w(0) = w_0, \tag{29.8}$$

where $-\mathcal{A}$ generates a uniformly bounded C_0-group $G(t), -\infty < t < +\infty$, on the Banach space X. We have assumed the presence of a group instead of a semigroup as our attention will now be turned to hyperbolic problems.

We assume the usual regularity conditions:

- f is continuously differentiable and uniformly bounded on $\bar{D} \times [0, \infty) \times [0, \varepsilon_0]$, where D is an open, bounded set in X.
- f can be expanded with respect to ε in a Taylor series, at least to some order.

The group $G(t)$ generates a generalized solution of equation (29.8) as a solution of the integral equation

$$w(t) = G(t)w_0 + \varepsilon \int_0^t G(t-s) f(w(s), s, \varepsilon) ds.$$

Using the variation of constants transformation $w(t) = G(t)z(t)$ for equation (29.8), we find what is called the standard form (see [SaVeMu07] or [Ve05])

$$\frac{dz}{dt} = \varepsilon F(z, s, \varepsilon), \; F(z, s, \varepsilon) = G(-s) f(G(s)z, s, \varepsilon). \tag{29.9}$$

In what follows we assume that $F(z, s, \varepsilon)$ is an almost periodic function in a Banach space, satisfying Bochner's criterion; see, for instance, [Ve05]. The average F^0 is defined by

$$F^0(z) = \lim_{T \to \infty} \frac{1}{T} \int_0^T F(z, s, 0) ds. \tag{29.10}$$

Applying normalization by the averaging transformation

$$z(t) = v(t) + \varepsilon \int_0^t \left(F(v, s, 0) - F^0(v) \right) ds, \; v(0) = w_0, \tag{29.11}$$

produces the normal form equation

$$\frac{dv}{dt} = \varepsilon F^0(v) + O(\varepsilon^2)$$

with the $O(\varepsilon^2)$ term still time dependent. There are at least two problems here: the generalized Fourier spectrum of the almost periodic function F contains an infinite number of frequencies, and the integral in equation (29.11) may not be bounded for all time, as is the case for periodic functions.

29.3.1 Averaging Theorem

The averaging approximation $\bar{z}(t)$ of $z(t)$ is obtained by omitting the $O(\varepsilon^2)$ terms:

$$\frac{d\bar{z}}{dt} = \varepsilon F^0(\bar{z}), \ \bar{z}(0) = w_0. \tag{29.12}$$

Under these rather general conditions, [Bu93] (or [Ve05]) provides the following theorem.

Theorem 1 (general averaging). *Consider equation (29.8) and the corresponding $z(t)$, $\bar{z}(t)$ given by equations (29.9) and (29.12) under the basic conditions stated above. If $G(t)\bar{z}(t)$ exists in an interior subset of D on the timescale $1/\varepsilon$, we have $v(t) - \bar{z}(t) = o(1)$ and*

$$z(t) - \bar{z}(t) = o(1) \,\text{as}\, \varepsilon \to 0$$

on the timescale $1/\varepsilon$. If $F(z, t, 0)$ is periodic in t, the error is $O(\varepsilon)$.

29.3.2 Approximations for All Time

In the case of attraction, averaging–normalization leads to stronger approximation results. The results can be described as follows. Consider the initial value problem in a Banach space

$$\dot{x} = \varepsilon f(x, t), \ x(0) = x_0.$$

Suppose that we can average the vector field:

$$f^0(z) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(z, s) ds$$

and thus can consider the averaged equation

$$\dot{z} = \varepsilon f^0(z), \ z(0) = x_0.$$

We have the following result by Sanchez-Palencia ([Sa75] and [Sa76]).

Theorem 2. *Suppose that the vector fields f and f^0 are continuously differentiable and that $z = a$ is an asymptotically stable critical point (in linear approximation) of the averaged equation. If x_0 lies within the domain of attraction of a, we have*

$$x(t) - z(t) = o(1) \text{ as } \varepsilon \to 0$$

for $t \geq 0$. If the vector field f is periodic in t, the error is $O(\varepsilon)$ for all time.

29.4 Normal Forms for Hyperbolic Equations

A straightforward application is to consider semilinear initial value problems of hyperbolic type,

$$u_{tt} + Au = \varepsilon f(u, u_t, t, \varepsilon), \quad u(0) = u_0, u_t(0) = v_0, \tag{29.13}$$

where A is a positive, self-adjoint linear operator on a separable Hilbert space and f satisfies the basic conditions. In our applications later on, we will be concerned with the case that we have one space dimension and that for $\varepsilon = 0$ we have a linear, dispersive wave equation by choosing

$$Au = -u_{xx} + u.$$

To make the relation with equation (29.8) explicit, one writes $u_1 = u, u_2 = u_t$, and

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= u_2, \\ \frac{\partial u_2}{\partial t} &= -Au_1 + \varepsilon f(u_1, u_2, t, \varepsilon). \end{aligned}$$

One uses the operator (with eigenvalues and eigenfunctions) associated with this system.

In particular and to focus ideas, consider the case of the boundary conditions $u(0, t) = u(\pi, t) = 0$.

In this case, a suitable domain for the eigenfunctions is $\{u \in W^{1,2}(0, \pi) : u(0) = u(\pi) = 0\}$. Here $W^{1,2}(0, \pi)$ is the Sobolev space consisting of functions $u \in L_2(0, \pi)$ that have first order generalized derivatives in $L_2(0, \pi)$. The eigenvalues are $\lambda_n = \sqrt{n^2 + 1}, n = 1, 2, \ldots$ and the spectrum is nonresonant. The implication is that $F(z, s, 0)$ in expression (29.10) is almost periodic.

Assume now for equation (29.13) homogeneous Dirichlet conditions or homogeneous Neumann conditions. The denumerable eigenvalues in this case are $\lambda_n = \omega_n^2$ and the corresponding eigenfunctions $v_n(x)$. Substitution of the expansion

$$u(x, t) = \sum u_n(t) v_n(x)$$

into equation (29.13) and taking inner products with the eigenfunctions $v_n(x)$ produces the infinite set of coupled second order equations

$$\ddot{u}_n + \omega_n^2 u_n = \varepsilon F(\mathbf{u}, t, \varepsilon), \tag{29.14}$$

with $\mathbf{u}$ representing the infinite set $u_n, \dot{u}_n$ with $n = 1, 2, 3, \ldots$ in the Dirichlet case, $n = 0, 1, 2, \ldots$ in the Neumann case.

We shall discuss the procedure for a few examples. The variation of constants transformation, introduced in the preceding sections, considers the case of the infinite-dimensional system (29.14) the following form. The standard transformation $u_n, \dot{u}_n \to y_{n_1}, y_{n_2}$ of the form

$$\begin{aligned} u_n &= y_{n_1} \cos \omega_n t + \frac{y_{n_2}}{\omega_n} \sin \omega_n t, \\ \dot{u}_n &= -\omega_n y_{n_1} \sin \omega_n t + y_{n_2} \cos \omega_n t, \end{aligned}$$

is introduced in system (29.14), followed by averaging. An alternative transformation to the standard form, $u_n, \dot{u}_n \to r_n, \psi_n$, employs amplitude-phase coordinates:

$$u_n = r_n \cos(\omega_n t + \psi_n), \ \dot{u}_n = -r_n \omega_n \sin(\omega_n t + \psi_n). \tag{29.15}$$

In general, averaging leaves us with an infinite-dimensional system that may still be difficult to analyze. In principle, however, it is simpler and will admit analysis.

In our analysis of hyperbolic PDEs, we will be interested in the case where we have a resonance between a finite number of modes k and that the infinite number of other, nonresonant modes are attracted to a stationary solution. To fix ideas, assume that these stationary states correspond with the trivial solutions of the modes as will be the case in our examples. The attraction is produced by dissipation.

With these assumptions, we shall split system (29.14) into two subsystems, a finite-dimensional resonant system and an infinite-dimensional nonresonant system.

29.5 Linear Waves with Parametric Excitation

Consider the linear wave equation

$$u_{tt} - c^2 u_{xx} + \varepsilon^k \beta u_t + (\omega_0^2 + \varepsilon \gamma \phi(t))u = 0, \ t \geq 0, \ 0 < x < \pi, \tag{29.16}$$

with boundary conditions $u_x(0, t) = u_x(\pi, t) = 0$, small, periodic or almost periodic parametric excitation $\varepsilon \gamma \phi(t)$, and small damping ($\beta > 0$); also $\omega_0 > 0$. The positive parameter $k \in \mathbb{N}$ indicates the size of the damping. For $\varepsilon = 0$ the model reduces to the dispersive wave equation of Section 29.4. In [RaEtAl99] the experimental motivation for this model is discussed, for instance a line of coupled pendulums with vertical (parametric) forcing or the linearized behavior of water waves in a vertically forced channel. Related mechanical problems can be found in [SeMa03].

29.5.1 Modal Expansion

Using the eigenfunctions for the Neumann problem $v_n(x) = \cos nx$, and eigenvalues $\omega_n^2 = \omega_0^2 + n^2c^2,\ n = 0, 1, 2, \ldots$, we expand the solution as

$$u(x,t) = \sum_0^\infty u_n(t) \cos nx.$$

Taking L_2-inner products with $v_n(x)$ produces the infinite-dimensional system

$$\ddot{u}_n + \omega_n^2 u_n = -\varepsilon^k \beta \dot{u}_n - \varepsilon \gamma u_n \phi(t),\ n = 0, 1, 2, \ldots, \tag{29.17}$$

with suitable initial conditions. System (29.17) is fully equivalent to equation (29.16). Note that the normal mode solutions do satisfy system (29.17), enabling the existence of an infinite number of finite- and infinite-dimensional invariant manifolds of equation (29.16). One question that remains is on the overall dynamics, and another is on the dynamics within the invariant manifolds. We will consider a number of cases to illustrate the subtleties involved.

29.5.2 The Mathieu Case $\phi(t) = \cos 2t$, No Resonance

We will show that if no basic frequency of the unperturbed modes, determined by the eigenvalues ω_n^2, resonates with the parametric frequency, all solutions will decay to zero if ε is small enough. The explicit condition for nonresonance is that for $n = 0, 1, 2, \ldots$

$$\omega_n^2 (= \omega_0^2 + n^2 c^2) \neq m^2,\ m = 0, 1, 2, \ldots.$$

Assume $k = 1$.
In the case of nonresonance we have, after introducing variation of constants as in Section 29.4 by $u_n, \dot{u}_n \to y_{n_1}, y_{n_2}$, the averaged normal form

$$\dot{y}_{n_1} = -\frac{1}{2}\varepsilon\beta y_{n_1} + O(\varepsilon^2),\ \dot{y}_{n_2} = -\frac{1}{2}\varepsilon\beta y_{n_2} + O(\varepsilon^2),\ n = 0, 1, 2, \ldots.$$

The solutions decay to first order to the trivial solution. Omitting the $O(\varepsilon^2)$ terms, we obtain approximations for the solutions that are, according to Theorem 2, valid for all time. We have explicitly

$$\begin{aligned} u_n(t) &= e^{-\frac{1}{2}\varepsilon\beta t}\left(u_n(0)\cos\omega_n t + \frac{\dot{u}_n(0)}{\omega_n}\sin\omega_n t\right) + o(1), \\ \dot{u}_n(t) &= e^{-\frac{1}{2}\varepsilon\beta t}\left(-u_n(0)\omega_n \sin\omega_n t + \dot{u}_n(0)\cos\omega_n t\right) + o(1), \end{aligned}$$

$n = 0, 1, 2, \ldots$, with the estimates $o(1)$ as $\varepsilon \to 0$ and *validity of the estimates for all positive time* ($t \geq 0$). For the energy of the modes of the system we have

$$E_n(t) = \frac{1}{2}(\dot{u}_n^2(t) + \omega_n^2 u_n^2(t)) = E_n(0)e^{-\varepsilon\beta t} + o(1)$$

for all time. This agrees with the standard theory for Mathieu equations.

What happens if the damping is smaller, $k > 1$? In this case we have to perform higher order averaging, to $O(\varepsilon^k)$. The results are qualitatively the same, but the attraction takes place on a longer timescale.

29.5.3 The Mathieu Case $\phi(t) = \cos 2t$, One Floquet Resonance

A nontrivial case arises if one of the eigenvalues equals 1 or is ε-close to it (this is called the first Floquet resonance), and there are no other accidental resonances. Suppose that $\omega_m^2 = 1 + \varepsilon d$, $m \neq 0$ and $k = 1$. The parameter d indicates the detuning from the resonance. Using averaging–normalization in amplitude-phase variables (29.15), we find after averaging, with some abuse of notation using the same r_n, ψ_n for the variables,

$$\begin{aligned}
\dot{r}_n &= -\varepsilon\frac{\beta}{2}r_n + O(\varepsilon^2), \ \ n \neq m, \\
\dot{\psi}_n &= O(\varepsilon^2), \ \ n \neq m, \\
\dot{r}_m &= \frac{1}{2}\varepsilon r_m(-\beta + \frac{\gamma}{2}\sin 2\psi_m) + O(\varepsilon^2), \\
\dot{\psi}_m &= \frac{1}{2}\varepsilon(d + \frac{\gamma}{2}\cos 2\psi_m) + O(\varepsilon^2) \ (m \neq 0).
\end{aligned}$$

The solution decays to the trivial solution if $\beta > |\gamma|/2$ (damping exceeds excitation). Suppose now that $2\beta/|\gamma| < 1$ with two solutions for ψ_m from

$$\sin 2\psi_m = \frac{2\beta}{\gamma}.$$

This value of ψ_m corresponds with a periodic solution if also

$$d + \frac{\gamma}{2}\cos 2\psi_m = 0.$$

This produces the condition

$$\beta^2 + d^2 = \frac{\gamma^2}{4},$$

representing the first order approximation to the well-known Floquet instability tongue in parameter space.

29.5.4 The Case of Quasi-Periodic Resonance

As we have started with an infinite-dimensional system, there is no end to the complications that may arise. Take, for instance, the case of a spectrum containing the first Floquet resonance $\omega_m = 1$ and a detuned higher order resonance, for instance $\omega_j = 4 + \delta(\varepsilon)d$. There are no other resonances.

In this case, all except two modes decay to a neighborhood of the trivial solution. The two remaining modes are described by

$$\begin{aligned} \ddot{u}_m + \omega_m^2 u_m &= -\varepsilon^k \beta \dot{u}_m - \varepsilon\gamma u_m \phi(t), \\ \ddot{u}_j + \omega_j^2 u_j &= -\varepsilon^k \beta \dot{u}_j - \varepsilon\gamma u_j \phi(t). \end{aligned}$$

The analysis again follows finite-dimensional Floquet theory, and this decoupling is in fact typical for the linear parametric wave equation. For a survey of perturbation methods for such parametric resonance problems, see [Ve09].

29.6 Nonlinear Waves with Parametric Excitation

Consider the wave equation

$$u_{tt} - c^2 u_{xx} + \varepsilon\beta u_t + (\omega_0^2 + \varepsilon\gamma\cos 2t)u = \varepsilon(au^2 + bu^3), \; t \geq 0, 0 < x < \pi, \quad (29.18)$$

with boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$, small, periodic parametric excitation $\varepsilon\gamma\cos 2t$, and small damping ($\beta > 0$); also $\omega_0 > 0$. For $\varepsilon = 0$ the model reduces again to the dispersive wave equation of Section 29.4.

In contrast to the case of a linear PDE, we now expect modal interactions. It turns out, surprisingly enough, that this is generally not the case.

29.6.1 Modal Expansion

Using as before the eigenfunctions for the Neumann problem $v_n(x) = \cos nx$, and eigenvalues $\omega_n^2 = \omega_0^2 + n^2c^2, \; n = 0, 1, 2, \ldots$, we expand the solution as

$$u(x,t) = \sum_0^\infty u_n(t)\cos nx.$$

Taking L_2-inner products with $v_n(x)$ produces the infinite-dimensional system

$$\ddot{u}_n + \omega_n^2 u_n = -\varepsilon\beta\dot{u}_n - \varepsilon\gamma u_n \cos 2t + \varepsilon f_n(\mathbf{u}), \; n = 0, 1, 2, \ldots, \quad (29.19)$$

with suitable initial conditions; $\mathbf{u} = (u_0, u_1, u_2, \ldots)$. The nonlinear terms are quadratic and cubic with constant coefficients.

System (29.19) is fully equivalent to equation (29.18). Note that the normal mode solutions *do not* satisfy system (29.19), so we do not have a priori normal mode invariant manifolds of equation (29.18). We will distinguish between the following cases:

- Wave speed and dispersion parameter c and ω_0 are $O(1)$ quantities with respect to ε.

- The wave speed c is $O(\varepsilon)$. In this case we have, assuming that ω_0 is an $O(1)$ quantity, for a finite number of modes the $1:1:1:\cdots$-resonance. This case has been discussed in [BMV].
- The dispersion is small: $\omega_0 = O(\varepsilon)$. In this case the system (29.19) is fully resonant. This problem is unsolved; see, for instance, the discussion in [Ve05].

29.6.2 Averaging–Normalization

Assuming that c and ω_0 are $O(1)$ quantities with respect to ε, we will carry out the averaging process. The fact that the spatial dimension is 1 means that all eigenvalues are single; this simplifies the averaging–normalization.

29.6.3 One Floquet Resonance

Assume that one of the eigenvalues is near resonant with respect to parametric excitation, for instance,

$$\omega_0^2 = 1 + \varepsilon d,$$

with d the detuning. The equations of motion become for $n = 0, 1, 2, \ldots$

$$\ddot{u}_n + (1 + n^2c^2)u_n = -\varepsilon(du_n + \beta\dot{u}_n + \gamma u_n \cos 2t) + \varepsilon f_n(\mathbf{u}). \tag{29.20}$$

Assume that there are no other resonances between the frequencies ω_n. Introducing again amplitude-phase variables (29.15), we find after averaging, with some abuse of notation using the same r_n, ψ_n for the variables,

$$\begin{aligned}
\dot{r}_0 &= \frac{1}{2}\varepsilon r_0(-\beta + \frac{1}{2}\gamma \sin 2\psi_0),\\
\dot{\psi}_0 &= \frac{1}{2}\varepsilon(d + \frac{1}{2}\gamma\cos 2\psi_0 - \frac{3}{4}br_0^2 - \frac{3}{4}b\sum_{k=1}^{\infty} r_k^2),\\
\dot{r}_n &= -\frac{1}{2}\varepsilon\beta r_n,\ n = 1, 2, \ldots,\\
\dot{\psi}_n &= \varepsilon b h_n(\mathbf{u}).
\end{aligned}$$

The right-hand sides h_n are quadratic in $u_0, u_1, \ldots$. The modes $n = 1, 2, \ldots$ are exponentially decreasing, nontrivial behavior can take place in mode 0 governed by

$$\begin{aligned}
\dot{r}_0 &= \frac{1}{2}\varepsilon r_0(-\beta + \frac{1}{2}\gamma \sin 2\psi_0),\\
\dot{\psi}_0 &= \frac{1}{2}\varepsilon(d + \frac{1}{2}\gamma\cos 2\psi_0 - \frac{3}{4}br_0^2).
\end{aligned}$$

For a critical point to exist, we have the condition (as in Subsection 29.5.3)

$$2\beta/|\gamma| < 1.$$

The solution decays to the trivial solution if $\beta > |\gamma|/2$ (damping exceeds excitation). Suppose now that we have solutions for ψ_m from

$$\sin 2\psi_m = \frac{2\beta}{\gamma}.$$

This critical value of ψ_m corresponds with a periodic solution if also

$$d + \frac{1}{2}\gamma \cos 2\psi_0 - \frac{3}{4} b r_0^2 = 0.$$

This is a different situation from the linear case discussed earlier, as this condition also determines r_0. Suppose we find a positive solution for r_0^2. For the eigenvalues of the critical point we find

$$\lambda_{1,2} = -\beta \pm \sqrt{5\beta^2 - \gamma^2 - 2d\gamma \cos 2\psi_0}.$$

From the existence condition we have $\gamma^2 > 4\beta^2$, so at exact Floquet resonance ($d = 0$), we have stability of the periodic solution. If $4\beta^2 < \gamma^2 < 5\beta^2$, the critical point is a node, if $\gamma^2 > 5\beta^2$, the critical point is a focus, and around the stable periodic solution the solutions are spiralling in.

The picture changes if $d \neq 0$ and large enough.

29.6.4 Additional Low Order Resonances

Assuming we have the 1 : 2 parametric resonance in mode 0, the conditions for a combined low order resonance in system (29.20) are

$$\frac{1}{1 + m^2c^2} = \frac{1}{4}, \frac{1}{9},$$

for certain mode m. We find, respectively, $m^2c^2 = 3$ and $m^2c^2 = 8$. These choices produce a 1 : 2- and a 1 : 3-resonance, respectively.

Analysis of the possibility of a first or second order resonance in three degrees of freedom according to the resonance classification in [SaVeMu07] produces no positive results, so we will consider two degrees of freedom only. It is no restriction to choose $m = 1$, and we will have three frequencies: ω_0, ω_1, and the frequency of parametric excitation 2.

29.6.5 Combined Floquet and 1 : 2-Resonance

We assume

$$\omega_0^2 = 1 + \varepsilon d_1,\ c^2 = 3 + \varepsilon(d_2 - d_1),\ \omega_1^2 = 4 + \varepsilon d_2,$$

with d_1, d_2 indicating the detunings of the three frequencies. The equations of motion from system (29.20) which may show modal interaction become

$$\begin{aligned}\ddot{u}_0 + \omega_0^2 u_0 &= -\varepsilon\beta\dot{u}_0 - \varepsilon\gamma u_0 \cos 2t + \varepsilon a(u_0^2 + \frac{1}{2}u_1^2) + \varepsilon b u_0(u_0^2 + \frac{3}{2}u_1^2),\\ \ddot{u}_1 + \omega_1^2 u_1 &= -\varepsilon\beta\dot{u}_1 - \varepsilon\gamma u_1 \cos 2t + \varepsilon a 2u_0u_1 + \varepsilon b u_1(3u_0^2 + \frac{3}{4}u_1^2).\end{aligned}$$

We find after averaging, using the same r_n, ψ_n for the variables,

$$\begin{aligned}\dot{r}_0 &= \frac{1}{2}\varepsilon r_0(-\beta + \frac{1}{2}\gamma\sin 2\psi_0),\\ \dot{\psi}_0 &= \frac{1}{2}\varepsilon(d_1 + \frac{1}{2}\gamma\cos 2\psi_0 - \frac{3}{4}br_0^2 - \frac{3}{4}br_1^2),\\ \dot{r}_1 &= -\frac{1}{2}\varepsilon\beta r_1,\\ \dot{\psi}_1 &= \varepsilon\frac{1}{4}(d_2 - \frac{1}{2}b(3r_0^2 + \frac{9}{8}r_1^2)).\end{aligned}$$

We conclude that, because of symmetry in the equations of motion, the 1 : 2-resonance is degenerate in this case. This symmetry degeneration is described in detail in [TuVe00].

29.6.6 Combined Floquet and 1 : 3-Resonance

We can repeat the analysis, assuming

$$\omega_0^2 = 1 + \varepsilon d_1,\ c^2 = 8 + \varepsilon(d_2 - d_1),\ \omega_1^2 = 9 + \varepsilon d_2.$$

As for the 1 : 2-resonance, we find that the 1 : 3-resonance in this case is degenerate because of symmetry. The only active resonance for system (29.20) takes place in mode 0.

29.7 Discussion

1. We conclude that after an interval of time, asymptotically larger than $1/\varepsilon$ (for instance $1/\varepsilon^2$), the right-hand sides of the infinite-dimensional, nonresonant systems which we encountered in Sections 29.5 and 29.6 become $o(1)$. Starting with $o(1)$ initial conditions, the nonresonant modes remain $o(1)$.
2. The manifold where the fast dynamics takes place is almost invariant. We conjecture that very small fluctuations are possible for the higher order modes, arising from the presence of higher order resonance manifolds containing stable and unstable periodic solutions with corresponding intersecting stable and unstable manifolds. These resonance manifolds are of very small size, and the analysis to describe them is subtle. For an analysis of such resonance manifolds in two-degree-of-freedom Hamiltonian systems, see [TuVe00].
   A related discussion, for a different PDE, can be found in [WiHo97].

3. The parametrically excited wave equation with dispersion and wave speed independent of ε displays a remarkable reduction to low dimensional (one mode) behavior. This becomes clear by averaging–normalization. The equation is also of practical interest; applications are cited in [RaEtAl99]. A number of the phenomena we found, periodic and quasi-periodic solutions, are stable and in this way open for experimental investigation.

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30

Internal Boundary Variations and Discontinuous Transversality Conditions in Mechanics

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30.1 Introduction

The aim of the analysis is to recover the impact equation and the jump in the total energy of a Lagrangian system over an impact from the stationarity conditions of a modified action integral. The analysis is accomplished by introducing internal boundary variations and thereby obtaining discontinuous transversality conditions as the stationarity conditions of the impulsive action integral. An impact in mechanics is defined as a discontinuity in the generalized velocities of a mechanical system which is induced by some impulsive forces. An interaction with some constraints may result in an impact and give rise to impulsive forces. The instant of impulsive action where a discontinuity in the generalized velocities occurs is considered as an internal boundary in the time domain. The consideration of certain types of variations at the internal boundaries, which are called internal boundary variations by the author, give rise to discontinuous transversality conditions. By introducing a boundary at an instant of a discontinuity, one has to notice that it has a bilateral character, in the sense that the boundary constitutes an upper boundary for one segment of the interval, whereas for the other segment it constitutes a lower boundary in the time domain. The constraints are therefore introduced symmetrically with respect to pre-impact and post-impact states. It is shown that the impact equation and the energy balance over an impact can be obtained in the form of stationarity conditions for the general impact case by applying the discontinuous transversality conditions. The stationarity conditions are obtained by the application of subdifferential calculus techniques to a suitable extended-valued lower semi-continuous generalized Bolza functional, which in this case is the impulsive action integral, that is evaluated on multiple intervals.

In the book [Br96] and the references cited therein, a thorough overview of impact mechanics is provided. The variations at the boundaries drew attention, especially in optimal control. It has been shown in [HaSe83] that time transversality conditions are independent of the other maximum princi-

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ple conditions. The cited reference clarifies some issues dealing with the necessary condition for the optimal terminal time in free terminal time optimal control problems. An early attempt to relate discontinuities in the generalized velocities in the framework of distribution theory is given in [BaAn72]. The distributional Euler equations are shown to recover the Weierstrass–Erdmann conditions. However, it remains to be clarified whether Weierstrass–Erdmann corner conditions are a suitable means to analyze variational problems with discontinuity in the state. However, in [BaAn72] no reference is made to impulsive interactions of the Lagrangian system with constraints. The concepts of internal boundary variations and discontinuous transversality conditions are developed by the author and are presented and discussed in [Yu07] and [Yu08a] with applications to optimal control. A characterization of these concepts in terms of upper and lower subderivatives to the extended-valued lower-semicontinuous value functional under several more general regularity assumptions can be found in [Yu08b].

30.2 Preliminaries

Let q, $\dot{q}$, $\ddot{q}$ represent the generalized position, velocity, and acceleration in the generalized coordinates of a scleronomic Lagrangian system with n degrees of freedom, respectively. Hamilton postulated in 1835 that if a Lagrangian system occupies certain positions at fixed times t_0 and t_f, then it should move between these two positions along those admissible trajectories $q(t) \in \mathcal{C}_n^1[t_0,\, t_f]$ which make the action integral

$$J(q(t), \dot{q}(t)) = \int_{t_0}^{t_f} L(q(s), \dot{q}(s))\, ds$$

stationary. The integrand $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called the Lagrangian and is defined as $L = T - U$, where $U(q)$ and $T(q(t), \dot{q}(t))$ represent the potential and kinetic energy, respectively. The stationarity conditions state that along an admissible trajectory the following Euler–Lagrange equations have to be fulfilled:

$$\frac{d}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j}, \quad j = 1, 2, \ldots, n. \tag{30.1}$$

In order to extend this analysis so that it can encompass Lagrangian systems subject to impacts, the search space for the admissible trajectories $q(t)$ needs to be extended from the space of continuously differentiable functions to the space of absolutely continuous functions $\mathcal{AC}_n[t_0,\, t_f]$. The generalized velocities $\dot{q}(t)$ become elements of the space of bounded variation functions $\mathcal{BV}_n[t_0,\, t_f]$. Functions of bounded variation, like the generalized velocities $\dot{q}$ of a mechanical system which is subject to impulsive forces, are associated with an $\mathbb{R}^n$-valued regular Borel measure $d\dot{q}$ on $[t_0, t_f]$:

$$d\dot{q} = \ddot{q}\, dt + \chi' d\sigma.$$

The absolutely continuous part of the measure $d\dot{q}$ is denoted by $\ddot{q}\,dt$. The Radon–Nikodym derivative of $d\dot{q}$ with respect to $d\sigma$ is given by χ' and $d\sigma$ is some regular Borel measure. The atoms of $d\dot{q}$ occur only at discontinuities of $\dot{q}$, of which there are at most countably many. Since the jumps of generalized velocities are induced by impulsive forces, these impulsive forces also occur at Lebesgue-negligible atoms and are countably many. The quantity $\Delta\dot{q}(t) = \dot{q}^+(t) - \dot{q}^-(t)$ is called the jump of the arc $\dot{q}$ at t, and if it is nonzero, then there is an atom of $d\dot{q}$ at t with this value. A given $\mathbb{R}^m$-valued function $g(q)$ ($f(q) = -g(q)$) represents the shortest distances between the Lagrangian system and the constraints, and these distances are always nonnegative (nonpositive) due to the impenetrability assumption. It is assumed that $D = \nabla_q f(q)$ has full rank. Further, the distances are formulated in the inertial coordinate frame, and the contacts are assumed to be perfect contacts without any friction interaction. If an impact occurs, then the conservation of momentum requires

$$\left(\frac{\partial L}{\partial \dot{q}_j}\right)^+ - \left(\frac{\partial L}{\partial \dot{q}_j}\right)^- = \langle d_j, \Gamma\rangle, \quad j = 1, 2, \ldots, n, \tag{30.2}$$

to hold. It states that the change in generalized momentum is equal to the generalized impulse ΓD. It is obtained by the Lebesgues–Stieltjes integration of the Euler–Lagrange equations over an impact time of measure zero. Here d_j denotes the jth column of the linear operator D. The operation $\langle\cdot,\cdot\rangle$ is the dual pairing of its arguments. Physically, the contact impulse is repelling and is therefore sign restricted. The contact impulse and the distance fulfill among others the following complementarity relation:

$$f_i(q) \leq 0, \quad \Gamma_i \geq 0, \quad f_i(q)\cdot\Gamma_i = 0, \quad i = 1, \ldots, m. \tag{30.3}$$

The total energy of the scleronomic Lagrangian system is given by its Hamiltonian:

$$H(q, \dot{q}) = T(q, \dot{q}) + V(q).$$

The differential measure of the Hamiltonian is given by

$$dH(q, \dot{q}) = \frac{dH}{dt}\,dt + \left(T^+ - T^-\right)\,d\sigma.$$

The Lebesgue–Stieltjes integration of the differential measure of the Hamiltonian over the impact time yields

$$\int_{\{t_{\mathrm{imp}}\}} dH = H^+ - H^- = T^+ - T^- = L^+ - L^-. \tag{30.4}$$

The latter equality in (30.4) is due to the fact that the potential energy U remains unaltered during an impact, since it only depends on the generalized positions, which remain constant over an impact. The difference $T^+ - T^-$ is nonzero if and only if there is an impulsive action that induces a jump in the generalized velocities. The Borel measurable part of the Hamiltonian H is therefore related to the jump in the kinetic energy in the following manner:

$$\begin{aligned} T\left(q(t^+_{\text{imp}}), \dot{q}(t^+_{\text{imp}})\right) - T\left(q(t^-_{\text{imp}}), \dot{q}(t^-_{\text{imp}})\right) \\ = \frac{1}{2}\left\langle \dot{q}(t^+_{\text{imp}}) + \dot{q}(t^-_{\text{imp}}), M(q(t_{\text{imp}}))\left(\dot{q}(t^+_{\text{imp}}) - \dot{q}(t^-_{\text{imp}})\right)\right\rangle \\ = \frac{1}{2}\left\langle \dot{q}(t^+_{\text{imp}}) + \dot{q}(t^-_{\text{imp}}), \Gamma D\right\rangle . \end{aligned} \tag{30.5}$$

The latter equality arises by inserting the expression (30.2) for the momentum balance over an impact in the energy balance (30.5). Here the linear operator $M(q(t))$, which is positive definite and symmetric, is defined element-wise as

$$m_{\text{ij}}(q(t)) = \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n.$$

30.3 Internal Boundary Variations and Discontinuous Transversality Conditions

The assumptions during an impact are given as follows:
Assumptions A

1. The generalized positions remain unaffected at the impact.
2. The impact happens during an atomic time instant t_{imp} of which there are at most countably many.
3. There are no impacts at initial time t_0 and final time t_f.

These assumptions are converted to requirements to the variations at the internal boundaries. Since the impact time t_{imp} is free, the bilateral character of the variations at pre-impact and post-impact states is also dependent on the variations of the impact time. Several families of variational curves which are parameterized by ϵ are introduced in order to generate the variations:

$$\begin{aligned} q(t,\epsilon) &= q(t) + \epsilon\, \hat{q}(t) = q(t) + \delta\, q(t), \\ \dot{q}(t,\epsilon) &= \dot{q}(t) + \epsilon\, \hat{\dot{q}}(t) = \dot{q}(t) + \delta\, \dot{q}(t), \\ q(t^+_{\text{imp}},\epsilon) &= q(t^+_{\text{imp}}) + \epsilon\, \hat{q}(t^+_{\text{imp}}) = q(t^+_{\text{imp}}) + \delta\, q(t^+_{\text{imp}}), \\ q(t^-_{\text{imp}},\epsilon) &= q(t^-_{\text{imp}}) + \epsilon\, \hat{q}(t^-_{\text{imp}}) = q(t^-_{\text{imp}}) + \delta\, q(t^-_{\text{imp}}), \\ \dot{q}(t^+_{\text{imp}},\epsilon) &= \dot{q}(t^+_{\text{imp}}) + \epsilon\, \hat{\dot{q}}(t^+_{\text{imp}}) = \dot{q}(t^+_{\text{imp}}) + \delta\, \dot{q}(t^+_{\text{imp}}), \\ \dot{q}(t^-_{\text{imp}},\epsilon) &= \dot{q}(t^-_{\text{imp}}) + \epsilon\, \hat{\dot{q}}(t^-_{\text{imp}}) = \dot{q}(t^-_{\text{imp}}) + \delta\, \dot{q}(t^-_{\text{imp}}), \\ t^+_{\text{imp}}(\epsilon) &= t^+_{\text{imp}} + \epsilon\, \hat{t}^+_{\text{imp}} = t^+_{\text{imp}} + \delta\, t^+_{\text{imp}}, \\ t^-_{\text{imp}}(\epsilon) &= t^-_{\text{imp}} + \epsilon\, \hat{t}^-_{\text{imp}} = t^-_{\text{imp}} + \delta\, t^-_{\text{imp}}. \end{aligned}$$

The variations of the pre- and post-impact generalized positions and velocities at fixed time $\hat{q}(t^+_{\text{imp}})$, $\hat{\dot{q}}(t^+_{\text{imp}})$, $\hat{q}(t^-_{\text{imp}})$, $\hat{\dot{q}}(t^-_{\text{imp}})$ are related to the total variations in these entities $\hat{q}^+_{\rm imp}$, $\hat{\dot{q}}^+_{\rm imp}$, $\hat{q}^-_{\rm imp}$, $\hat{\dot{q}}^-_{\rm imp}$ at $t^+_{\rm imp}$ and $t^-_{\rm imp}$ by the following affine relations:

$$\hat{q}(t^+_{\rm imp}) = \hat{q}^+_{\rm imp} - \dot{q}(t^+_{\rm imp})\ \hat{t}^+_{\rm imp}, \tag{30.6}$$

$$\hat{q}(t^-_{\rm imp}) = \hat{q}^-_{\rm imp} - \dot{q}(t^-_{\rm imp})\ \hat{t}^-_{\rm imp}, \tag{30.7}$$

$$\hat{\dot{q}}(t^+_{\rm imp}) = \hat{\dot{q}}^+_{\rm imp} - \ddot{q}(t^+_{\rm imp})\ \hat{t}^+_{\rm imp} - \hat{\chi}^+_{\rm imp}, \tag{30.8}$$

$$\hat{\dot{q}}(t^-_{\rm imp}) = \hat{\dot{q}}^-_{\rm imp} - \ddot{q}(t^-_{\rm imp})\ \hat{t}^-_{\rm imp} - \hat{\chi}^-_{\rm imp}. \tag{30.9}$$

By considering the affine relations given in equations (30.6) to (30.9) the boundary variations are decomposed into orthogonal independent variations in $\hat{t}^-_{\rm imp}$, $\hat{t}^+_{\rm imp}$, $\hat{\dot{q}}(t^+_{\rm imp})$, $\hat{\dot{q}}(t^-_{\rm imp})$, $\hat{q}(t^+_{\rm imp})$, $\hat{q}(t^-_{\rm imp})$ at the impact. In determining the stationarity conditions of the impulsive action integral, the internal boundary variations at the impact are given by the finite-dimensional set $\hat{\mathcal{V}}$:

$$\hat{\mathcal{V}} = \left\{\hat{t}^-_{\rm imp},\ \hat{t}^+_{\rm imp},\ \hat{\dot{q}}(t^+_{\rm imp}),\ \hat{\dot{q}}(t^-_{\rm imp}),\ \hat{q}(t^+_{\rm imp}),\ \hat{q}(t^-_{\rm imp})\right\} \subseteq \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n.$$

Having set the stage, the impulsive action integral is stated as

$$J\left(q(t),\ \dot{q}(t),\ t_{\rm imp}\right) = \int_{t_0}^{t^-_{\rm imp}} L(q(s),\dot{q}(s))\,ds + \int_{t^+_{\rm imp}}^{t_f} L(q(s),\dot{q}(s))\,ds + \Psi_{\mathcal{C}^+_{\rm imp}} + \Psi_{\mathcal{C}^-_{\rm imp}}. \tag{30.10}$$

The initial and final times t_0 and t_f are fixed. The sets $\mathcal{C}^+_{\rm imp}$ and $\mathcal{C}^-_{\rm imp}$ are defined as

$$\mathcal{C}^+_{\rm imp} = \left\{\{q(t_{\rm imp}), t_{\rm imp}\}\,|\, f(q(t^+_{\rm imp})) \le \mathbf{0},\, f(q(t^+_{\rm imp})) \in \mathcal{C}^m_1\right\}, \tag{30.11}$$

$$\mathcal{C}^-_{\rm imp} = \left\{\{q(t_{\rm imp}), t_{\rm imp}\}\,|\, f(q(t^-_{\rm imp})) \le \mathbf{0},\, f(q(t^-_{\rm imp})) \in \mathcal{C}^m_1\right\}. \tag{30.12}$$

The contact durations of the Lagrangian system with the boundary of the constraint manifolds $\partial\mathcal{C}^+_{\rm imp}$ and $\partial\mathcal{C}^-_{\rm imp}$ are assumed to have measure zero, so that only impulsive interactions are allowed. The indicator function $\Psi_{\mathcal{C}}(x)$ of a closed and compact set $\mathcal{C}$ takes the value zero if $x\in\mathcal{C}$ and infinity otherwise. Given the lower semi-continuous extended-value functional J, the stationarity condition requires that the lower subderivatives of the value functional $J^{\downarrow}(\cdot;\hat{\psi})$ are all nonnegative with respect to the admissible variations:

$$J^{\downarrow}(\cdot;\hat{\psi}) \ge 0, \quad \forall\,\hat{\psi}\in\hat{\mathcal{V}}\bigcup\left\{\hat{q}(t),\hat{\dot{q}}(t)\right\} \text{ and } \hat{\psi}\,\text{admissible}.$$

Functional J is directionally Lipschitzian in all directions $\hat{\psi}\in\hat{\mathcal{V}}\bigcup\left\{\hat{q}(t),\hat{\dot{q}}(t)\right\}$. By reverting to the definition of the upper and lower subderivative as given in the Appendix, one notices that the lower and upper subderivatives coincide in the directionally Lipschitzian case:

$$J^{\downarrow}(\cdot;\hat{\psi}) = J^{\uparrow}(\cdot;\hat{\psi}), \quad \forall\, \hat{\psi} \in \hat{\mathcal{V}} \bigcup \left\{ \hat{q}(t), \hat{\dot{q}}(t) \right\}.$$

In what follows, it is shown that the stationarity conditions of the functional (30.10) subject to constraints (30.11) and (30.12) recover the Euler–Lagrange equations (30.1), the impact equation (30.2), and the energy balance over an impact (30.5).

Indeed, if there exist trajectories $\tilde{q}(t)$ and $\tilde{\dot{q}}(t)$, impact position $\tilde{q}(\tilde{t}_{\mathrm{imp}})$, pre-impact and post-impact generalized velocities $\tilde{\dot{q}}(\tilde{t}^-_{\mathrm{imp}})$ and $\tilde{\dot{q}}(\tilde{t}^+_{\mathrm{imp}})$ at an impact time $\tilde{t}_{\mathrm{imp}}$, which all together make the Bolza functional in (30.10) stationary, such that the value function assumes the finite value $\tilde{J}(\epsilon = 0) = J\left(\tilde{q}(t),\, \tilde{\dot{q}}(t),\, \tilde{t}_{\mathrm{imp}}\right)$, then the following variational inequality is also fulfilled:

$$\liminf_{\epsilon \to 0^+} \frac{J(\epsilon) - \tilde{J}(0)}{\epsilon} = \sum_{\forall \hat{\psi}} \tilde{J}^{\uparrow}(\cdot, \hat{\psi}) \geq 0, \quad \forall \hat{\psi} \in \hat{\mathcal{V}} \bigcup \left\{ \hat{q}(t), \hat{\dot{q}}(t) \right\}, \tag{30.13}$$

since J is directionally Lipschitzian, lower semi-continuous, and subdifferentially regular at any stationary solution. Here $J(\epsilon)$ is an abbreviation for $J\big(\, q(t,\epsilon),\, \dot{q}(t,\epsilon),\, t^+_{\mathrm{imp}}(\epsilon),\, t^-_{\mathrm{imp}}(\epsilon),\, q(t^-_{\mathrm{imp}},\epsilon),\, q(t^+_{\mathrm{imp}},\epsilon),\, \dot{q}(t^+_{\mathrm{imp}},\epsilon),\, \dot{q}(t^-_{\mathrm{imp}},\epsilon)\big)$. The upper subderivative of J in the direction $\delta q(t)$ is given by

$$J^{\uparrow}\left(\cdot, \hat{q}(t)\right) = \int_{t_0}^{t^-_{\mathrm{imp}}} \frac{\partial L}{\partial q}\left(q(s), \dot{q}(s)\right) \delta q(s)\, ds + \int_{t^+_{\mathrm{imp}}}^{t_f} \frac{\partial L}{\partial q}\left(q(s), \dot{q}(s)\right) \delta q(s)\, ds.$$

The upper subderivative of J in the direction $\delta \dot{q}(t)$ is given by

$$J^{\uparrow}\left(\cdot, \hat{\dot{q}}(t)\right) = \int_{t_0}^{t^-_{\mathrm{imp}}} \frac{\partial L}{\partial \dot{q}}\left(q(s), \dot{q}(s)\right) \delta \dot{q}(s)\, ds + \int_{t^+_{\mathrm{imp}}}^{t_f} \frac{\partial L}{\partial \dot{q}}\left(q(s), \dot{q}(s)\right) \delta \dot{q}(s)\, ds. \tag{30.14}$$

After applying the du Bois–Reymond lemma twice in (30.14), this directional derivative can be related to the boundary variations $\delta q(t^+_{\mathrm{imp}})$, $\delta q(t^-_{\mathrm{imp}})$ and to the variation in generalized positions $\delta q(t)$ on the interior of a time domain $[a, b]$:

$$\int_a^b \frac{\partial L}{\partial \dot{q}}\, \delta \dot{q}(s)\, ds = \frac{\partial L}{\partial \dot{q}}\left(q(b), \dot{q}(b)\right) \delta q(b) - \frac{\partial L}{\partial \dot{q}}\left(q(a), \dot{q}(a)\right) \delta q(a) - \int_a^b \frac{d}{\mathrm{dt}} \frac{\partial L}{\partial \dot{q}}\, \delta q(s)\, ds.$$

The upper subderivative of J in the direction $\delta q(t^+_{\mathrm{imp}})$ then becomes

$$J^{\uparrow}\left(\cdot, \hat{q}(t^+_{\mathrm{imp}})\right) = \left(-\left(\frac{\partial L}{\partial \dot{q}}\right)^+ + \lambda^+ D \right) \delta q(t^+_{\mathrm{imp}}).$$

Similarly, the upper subderivative of J in the direction $\delta q(t^-_{\mathrm{imp}})$ can be stated as

$$J^{\uparrow}\left(\cdot, \hat{q}(t_{\text{imp}}^{-})\right) = \left(\left(\frac{\partial L}{\partial \dot{q}}\right)^{-} + \lambda^{-} D\right) \delta q(t_{\text{imp}}^{-}).$$

The vectors λ^{+} and λ^{-} are dual multipliers which are restrained to ${}_{0}^{+}\mathbb{R}^{1\times m}$. The upper subderivative of J in the direction $\delta t_{\text{imp}}^{-}$ is given by

$$J^{\uparrow}\left(\cdot, \hat{t}_{\text{imp}}^{+}\right) = \left(-L(q(t_{\text{imp}}^{+}), \dot{q}(t_{\text{imp}}^{+})) + \lambda^{+} D\, \dot{q}(t_{\text{imp}}^{+})\right) \delta t_{\text{imp}}^{+}. \tag{30.15}$$

The upper subderivative of J in the direction $\delta t_{\text{imp}}^{+}$ is given by

$$J^{\uparrow}\left(\cdot, \hat{t}_{\text{imp}}^{-}\right) = \left(L(q(t_{\text{imp}}^{-}), \dot{q}(t_{\text{imp}}^{-})) + \lambda^{-} D\, \dot{q}(t_{\text{imp}}^{-})\right) \delta t_{\text{imp}}^{-}. \tag{30.16}$$

As a result of the analysis, the following variational inequality (VI) is obtained:

$$\begin{aligned}
&\liminf_{\epsilon\to 0^{+}} \frac{J(\epsilon) - \tilde{J}(0)}{\epsilon} = \qquad (30.17)\\
&\left(\tilde{L}^{-} + \tilde{\lambda}^{-} \tilde{D}\tilde{\dot{q}}(\tilde{t}_{\text{imp}}^{-})\right) \delta t_{\text{imp}}^{-} + \int_{t_0}^{\tilde{t}_{\text{imp}}^{-}} \left[\frac{\partial L}{\partial q} - \frac{d}{\text{dt}}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] (\tilde{q}, \tilde{\dot{q}})\delta q(s)\, ds\\
+\ &\left(-\tilde{L}^{+} + \tilde{\lambda}^{+} \tilde{D}\, \tilde{\dot{q}}(\tilde{t}_{\text{imp}}^{+})\right) \delta t_{\text{imp}}^{+} + \int_{\tilde{t}_{\text{imp}}^{+}}^{t_f} \left[\frac{\partial L}{\partial q} - \frac{d}{\text{dt}}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] (\tilde{q}, \tilde{\dot{q}})\delta q(s)\, ds\\
+\ &\left(\tilde{\lambda}^{+} \tilde{D} - \left(\frac{\partial \tilde{L}}{\partial \dot{q}}\right)^{+}\right) \delta q(t_{\text{imp}}^{+}) + \left(\tilde{\lambda}^{-} \tilde{D} + \left(\frac{\partial \tilde{L}}{\partial \dot{q}}\right)^{-}\right) \delta q(t_{\text{imp}}^{-}) \geq 0.
\end{aligned}$$

Since all variations are independent of each other, the VI has to be fulfilled by every variational expression as stated in Theorem 2 in the Appendix. Since $\tilde{J}$ is subdifferentially regular, the fulfillment of each VI for each expression separately is equivalent to the fulfillment of the VI as cited in (30.17). By the application of the Lebesgue dominated convergence theorem on the integrals in (30.17), the Euler–Lagrange equations are obtained in the almost everywhere sense as given in (30.1). By assumption A.1 the variations of pre-impact position and post-impact position have to be of equal magnitude and direction, and therefore are not independent of each other, $\delta q(t_{\text{imp}}^{+}) = \delta q(t_{\text{imp}}^{-}) = \delta q(t_{\text{imp}})$.

By assumption A.2 the variations $\delta t_{\text{imp}}^{-}$ and $\delta t_{\text{imp}}^{+}$ have to be of equal magnitude and direction, and therefore are not independent of each other, $\delta t_{\text{imp}}^{+} = \delta t_{\text{imp}}^{-} = \delta t_{\text{imp}}$. By making use of the dependence of the impact position variations, one obtains

$$\left((\tilde{\lambda}^{+} + \tilde{\lambda}^{-})\, \tilde{D} - \left(\frac{\partial \tilde{L}}{\partial \dot{q}}\right)^{+} + \left(\frac{\partial \tilde{L}}{\partial \dot{q}}\right)^{-}\right) \delta q(t_{\text{imp}}) \geq 0. \tag{30.18}$$

By making use of the dependence of the time variations, the following VI is obtained:

$$\left(\tilde{L}^- + \tilde{\lambda}^- \, \tilde{D}\tilde{\dot{q}}(\tilde{t}^-_{\mathrm{imp}}) - \tilde{L}^+ + \tilde{\lambda}^+ \, \tilde{D} \, \tilde{\dot{q}}(\tilde{t}^+_{\mathrm{imp}})\right) \, \delta t_{\mathrm{imp}} \geq 0. \tag{30.19}$$

Since the variations $\delta q(t_{\mathrm{imp}})$ are unrestrained, the following is required in order for the VI (30.18) to hold:

$$\left(\frac{\partial \tilde{L}}{\partial \dot{q}}\right)^+ - \left(\frac{\partial \tilde{L}}{\partial \dot{q}}\right)^- = (\tilde{\lambda}^+ + \tilde{\lambda}^-) \, \tilde{D}. \tag{30.20}$$

Since the variations $\delta q(t_{\mathrm{imp}})$ are unrestrained, the following is required in order for the VI (30.19) to hold:

$$\tilde{L}^+ - \tilde{L}^- = \tilde{\lambda}^- \, \tilde{D}\tilde{\dot{q}}^T(\tilde{t}^-_{\mathrm{imp}}) + \tilde{\lambda}^+ \, \tilde{D} \, \tilde{\dot{q}}^T(\tilde{t}^+_{\mathrm{imp}}). \tag{30.21}$$

From the comparison of the equations given in (30.20) and (30.21) with the momentum balance (30.2) and the energy balance (30.5), the relations given in (30.22), (30.23) follow immediately:

$$\tilde{\lambda}^+_j + \tilde{\lambda}^-_j = \Gamma_j, \qquad j = 1, \ldots, m. \tag{30.22}$$

$$\tilde{\lambda}^+_j = \tilde{\lambda}^-_j = \frac{\Gamma_j}{2}, \qquad j = 1, \ldots, m. \tag{30.23}$$

The element-wise equality of the dual multipliers as stated in equations (30.23) means that the constraints given in (30.11) and (30.12) are, as expected due to the symmetrical attributes of the internal boundary, equally weighted in determining the stationarity conditions of the impulsive action integral. Further, each $\tilde{\lambda}^+_j$ and $\tilde{\lambda}^-_j$ is nonnegative, so the sign restriction of each Γ_j is enforced, which is stated in the complementarity relation (30.3).

30.4 Appendix

The proofs of the main theorems and more detailed discussions on various definitions in subdifferential calculus of extended-value functionals can be verified in references [Ro79], [Ro80], [Ro85], and [Ro04].

Definition 1 (upper and lower subderivatives). *Let f be any extended real-valued lower semi-continuous function on a linear topological space $\mathcal{E}$, and let x be any point where f is finite. The upper subderivative of f at x with respect to y is defined by*

$$f^{\uparrow}(x; y) = \limsup_{\substack{x' \to_f x \\ t \downarrow 0}} \; \inf_{y' \to y} \frac{f(x' + t\, y') - f(x')}{t}.$$

The lower subderivative of f at x with respect to y is defined by

$$f^{\downarrow}(x;y) = \liminf_{\substack{x' \underset{f}{\to} x \\ t \downarrow 0}} \sup_{y' \to y} \frac{f(x' + t\,y') - f(x')}{t},$$

where

$$x' \underset{f}{\to} x \Leftrightarrow \quad x' \Rightarrow x \quad \wedge \quad f(x') \Rightarrow f(x).$$

Theorem 1. *Let f be any extended-real valued function on a linear topological space $\mathcal{E}$, and let x be any point where f is finite. Then the "upper" subdifferential $\partial f(x)$ is a weak*-closed convex subset of $\mathcal{E}^*$ and*

$$\partial f(x) = \left\{ z \in \mathcal{E}^* \mid (z, -1) \in \mathcal{N}_{epi\,f}\,(x,\, f(x)) \right\} .$$

If $f^{\uparrow}(x;\,0) = -\infty$, then $\partial f(x)$ is empty, but otherwise $\partial f(x)$ is nonempty and

$$f^{\uparrow}(x;\, y) = \sup\, \{\, \langle\, y,\, z\rangle \mid z \in \partial f(x), \quad \forall y \in \mathcal{E}\}\ .$$

Analogously, the "lower" subdifferential $\tilde{\partial} f(x)$ is a weak*-closed convex subset of $\mathcal{E}^*$ and

$$\tilde{\partial} f(x) = \left\{ z \in \mathcal{E}^* \mid (z, -1) \in \mathcal{N}_{hypo\,f}\,(x,\, f(x)) \right\} .$$

If $f^{\downarrow}(x;\,0) = \infty$, then $\tilde{\partial} f(x)$ is empty, but otherwise $\tilde{\partial} f(x)$ is nonempty and

$$f^{\downarrow}(x;\, y) = \inf \left\{ \langle\, y,\, z\rangle \mid z \in \tilde{\partial} f(x), \quad \forall y \in \mathcal{E} \right\}.$$

Definition 2 (subdifferential regularity). *A function f is called subdifferentially regular at x if f is finite at x and*

$$\liminf_{\substack{y' \to y \\ t \downarrow 0}} \frac{f(x + t\,y) - f(x)}{t} = f^{\uparrow}(x;y), \quad \forall\, y.$$

Proposition 1. *Suppose that $\mathcal{C}$ is a smooth manifold around x in the sense that*

$$\mathcal{C} = \{x \,|\, g_j(x) = 0, \;\; for \quad j = 1, \ldots, r\},$$

where the functions g_j are continuously differentiable around y and the gradients $\nabla g_j(x)$, $j = 1, \ldots, r$ are linearly independent. Then $\mathcal{K}_{\mathcal{C}}(x)$ (contingent cone) is convex, and in fact

$$\mathcal{K}_{\mathcal{C}}(x) = \{y \,|\, \langle y,\, \nabla g_j(x)\rangle = 0, \quad for \;\; j = 1, \ldots, r\}.$$

Theorem 2. *Let f_1 and f_2 be extended real-valued functions on $\mathcal{E}$ that are finite at x. Suppose that f_2 is directionally Lipschitzian at x and*

$$\{y \,|\, f_1^{\uparrow}(x;y) < \infty\} \cap int\{y \,|\, f_2^{\uparrow}(x;y) < \infty\} \neq \emptyset\ .$$

Then

$$\begin{aligned}
(f_1 + f_2)^{\uparrow}(x; y) &\leq f_1^{\uparrow}(x; y) + f_2^{\uparrow}(x; y), \quad \forall y && (30.24)\\
\partial (f_1 + f_2)(x) &\subset \partial f_1(x) + \partial f_2(x)\ . && (30.25)
\end{aligned}$$

Equality holds in (30.25) if f_1 *and* f_2 *are also subdifferentially regular. It also holds in (30.24) if in addition* $f_1^{\uparrow}(x; y)$ *and* $f_2^{\uparrow}(x; y)$ *are not* $-\infty$ *(i.e.,* $\partial f_1(x)$ *and* $\partial f_2(x)$ *are nonempty), and in that event* f_1+f_2 *is likewise subdifferentially regular.*

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31

Regularization of Divergent Integrals in Boundary Integral Equations for Elastostatics

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31.1 Main Equations of Elastostatics

Let consider a homogeneous, linearly elastic body, which in three-dimensional (3-D) Euclidean space $\mathbb{R}^3$ occupies volume V with smooth boundary ∂V. The region V is an open bounded subset of the 3-D Euclidean space $\mathbb{R}^3$ with a $C^{0,1}$ Lipschitzian regular boundary ∂V. The boundary contains two parts ∂V_u and ∂V_p such that $\partial V_u \cap \partial V_p = \emptyset$ and $\partial V_u \cup \partial V_p = \partial V$. On the part ∂V_u are prescribed displacements $u_i(\mathbf{x})$ of the body points and on the part ∂V_p are prescribed tractions $p_i(\mathbf{x})$, respectively. The body may be affected by volume forces $b_i(\mathbf{x})$. We assume that displacements of the body points and their gradients are small, so its stress-strain state is described by the small strain deformation tensor $\varepsilon_{ij}(\mathbf{x})$. Then differential equations of equilibrium in the form of displacements may be presented in the form

$$A_{ij}u_j + b_i = 0, \quad A_{ij} = \mu\delta_{ij}\partial_k\partial_k + (\lambda+\mu)\partial_i\partial_j \quad \forall \mathbf{x} \in V, \tag{31.1}$$

where λ and μ are Lamé constants, $\mu > 0$ and $\lambda > -\mu$, and δ_{ij} is the Kronecker symbol.

If the problem is defined in an infinite region, then solution of the equations (31.1) must satisfy additional conditions at infinity in the form

$$u_j(\mathbf{x}) = O(r^{-1}), \quad \sigma_{ij}(\mathbf{x}) = O(r^{-2}) \quad \text{as } r \to \infty, \tag{31.2}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the distance in the 3-D Euclidean space.

If the body occupied a finite region V with the boundary ∂V, it is necessary to establish boundary conditions. We consider the mixed boundary conditions in the form

$$u_i(\mathbf{x}) = \varphi_i(\mathbf{x}) \quad \forall \mathbf{x} \in \partial V_u,$$
$$p_i(\mathbf{x}) = \sigma_{ij}(\mathbf{x})n_j(\mathbf{x}) = P_{ij}[u_j(\mathbf{x})] = \psi_i(\mathbf{x}) \quad \forall \mathbf{x} \in \partial V_p.$$

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The differential operator $P_{ij} : u_j \to p_i$ is called the stress operator. It transforms displacements into tractions. For homogeneous anisotropic and isotropic media they have the forms

$$P_{ij} = \lambda n_i \partial_k + \mu(\delta_{ij}\partial_n + n_k \partial_i),$$

respectively. Here n_i are components of the outward normal vector, and $\partial_n = n_i \partial_i$ is a derivative in the direction of the vector $\mathbf{n}(\mathbf{x})$ normal to the surface ∂V_p.

31.2 Integral Representations and Boundary Potentials

In order to establish integral representations for the displacements and tractions, let us consider Betti‘s reciprocal theorem,

$$\int_V (b_i u_i^* - b_i^* u_i) dV = \int_{\partial V} (p_i^* u_i - p_i u_i^*) dS. \tag{31.3}$$

This theorem is usually used to obtain integral representations for the displacements and traction vectors. To do that, we consider solutions of the elliptic partial differential equation (31.1) in an infinite space for the body force $b_i^*(\mathbf{x}) \to \delta_{ij}\delta(\mathbf{x} - \mathbf{y})$,

$$A_{ij} U_{kj}(\mathbf{x} - \mathbf{y}) + \delta_{ki}\delta(\mathbf{x} - \mathbf{y}) = 0 \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^3.$$

Solution of this equation have to satisfy conditions at infinity (31.2). Now considering that

$$u_i^*(\mathbf{x}) \to U_{ij}(\mathbf{x} - \mathbf{y}) \quad \text{and} \quad p_i^*(\mathbf{x}) \to P_{ij}[u_j^*(\mathbf{x})] = W_{ij}(\mathbf{x}, \mathbf{y}),$$

from (31.3) we obtain the integral representation for the displacements vector

$$u_i(\mathbf{y}) = \int_{\partial V} (p_j(\mathbf{x}) U_{ji}(\mathbf{x} - \mathbf{y}) - u_j(\mathbf{x}) W_{ji}(\mathbf{x}, \mathbf{y})) dS + \int_V p_j(\mathbf{x}) U_{ji}(\mathbf{x} - \mathbf{y}) dV, \tag{31.4}$$

which is called Somigliana’s formula. The kernels $U_{ji}(\mathbf{x} - \mathbf{y})$ and $W_{ji}(\mathbf{x}, \mathbf{y})$ are called fundamental solutions for elastostatics.

Applying to the differential operator P_{ij} (31.4), we will find integral representation for the traction in the form

$$p_j(\mathbf{y}) = \int_{\partial V} (p_j(\mathbf{x}) K_{ji}(\mathbf{x}, \mathbf{y}) - u_j(\mathbf{x}) F_{ji}(\mathbf{x}, \mathbf{y})) dS + \int_V p_j(\mathbf{x}) K_{ji}(\mathbf{x}, \mathbf{y}) dV. \tag{31.5}$$

The kernels $K_{ji}(\mathbf{x}, \mathbf{y})$ and $F_{ji}(\mathbf{x}, \mathbf{y})$ may be obtained by applying the differential operator P_{ij} to the kernels $U_{ji}(\mathbf{x} - \mathbf{y})$ and $W_{ji}(\mathbf{x}, \mathbf{y})$, respectively.

The integral representations (31.4) and (31.5) are usually used for direct formulation of the boundary integral equations in elastostatics.

Simple observation shows that kernels in the integral representations (31.4) and (31.5) tend to infinity when $r \to 0$. A more detailed analysis of the fundamental solutions gives us the following results [BaSlSl89], [Ba94], [MuMu05].

In the 3-D case with $\mathbf{x} \to \mathbf{y}$,

$$U_{ij}(\mathbf{x}-\mathbf{y}) \to r^{-1}, \;\; W_{ij}(\mathbf{x},\mathbf{y}) \to r^{-2}, \;\; K_{ij}(\mathbf{x},\mathbf{y}) \to r^{-2}, \;\; F_{ij}(\mathbf{x},\mathbf{y}) \to r^{-3}.$$

In the 2-D case with $\mathbf{x} \to \mathbf{y}$,

$$U_{ij}(\mathbf{x}-\mathbf{y}) \to \ln(r^{-1}), \; W_{ij}(\mathbf{x},\mathbf{y}) \to r^{-1}, \; K_{ij}(\mathbf{x},\mathbf{y}) \to r^{-1}, \; F_{ij}(\mathbf{x},\mathbf{y}) \to r^{-2}.$$

The integrals with singularities cannot be considered in the usual (Riemann or Lebesgue) sense. In order for such integrals to have sense, it is necessary to take special consideration of them. We will apply the following definitions of the integrals from (31.4) and (31.5).

Definition 1. *Integrals with kernels $U_{ij}(\boldsymbol{x}-\boldsymbol{y})$ are weakly singular (WS) and must be considered as improper*

$$W.S. \int\limits_{\partial V} p_i(\boldsymbol{x}) U_{ij}(\boldsymbol{x}-\boldsymbol{y}) dS = \lim_{\varepsilon \to 0} \int\limits_{\partial V \setminus \partial V_\varepsilon} p_i(\boldsymbol{x}) U_{ij}(\boldsymbol{x}-\boldsymbol{y}) dS.$$

Here ∂V_ε is a part of the boundary, the projection of which on the tangential plane is contained in the circle $C_\varepsilon(\boldsymbol{x})$ of radius ε with center at the point $\boldsymbol{x}$.

Definition 2. *Integrals with kernels $W_{ij}(\boldsymbol{x},\boldsymbol{y})$ and $K_{ij}(\boldsymbol{x},\boldsymbol{y})$ are singular and must be considered in the sense of the Cauchy principal values (PV) as*

$$P.V. \int\limits_{\partial V} u_i(\boldsymbol{x}) W_{ij}(\boldsymbol{x},\boldsymbol{y}) dS = \lim_{\varepsilon \to 0} \int\limits_{\partial V \setminus \partial V(r<\varepsilon)} u_i(\boldsymbol{x}) W_{ij}(\boldsymbol{x},\boldsymbol{y}) dS,$$

$$P.V. \int\limits_{\partial V} p_i(\boldsymbol{x}) K_{ij}(\boldsymbol{x},\boldsymbol{y}) dS = \lim_{\varepsilon \to 0} \int\limits_{\partial V \setminus \partial V(r<\varepsilon)} p_i(\boldsymbol{x}) K_{ij}(\boldsymbol{x},\boldsymbol{y}) dS.$$

Here $\partial V(r < \varepsilon)$ is a part of the boundary, the projection of which on the tangential plane is the circle $C_\varepsilon(\boldsymbol{x})$ of radius ε with center at the point $\boldsymbol{x}$.

Definition 3. *Integrals with kernels $F_{ij}(\boldsymbol{x},\boldsymbol{y})$ are hypersingular and must be considered in the sense of the Hadamard finite part (FP) as*

$$F.P. \int\limits_{\partial V} u_i(\boldsymbol{x}) F_{ji}(\boldsymbol{x},\boldsymbol{y}) dS = \lim_{\varepsilon \to 0} \left(\int\limits_{\partial V \setminus \partial V(r<\varepsilon)} u_i(\boldsymbol{x}) F_{ji}(\boldsymbol{x},\boldsymbol{y}) dS + 2u_j(\boldsymbol{x}) \frac{f_j(\boldsymbol{x})}{\partial V(r<\varepsilon)} \right).$$

Here functions $f_j(\boldsymbol{x})$ are chosen from the condition of the limit existence.

31.3 Regularization of Divergent Integrals

In order to solve the boundary integral equations numerically, we have to transform them to finite-dimensional equations. In order to do that transformation, we have to calculate integrals with various singularities. One of the main problems occurs in this situation is the presence of the divergent integrals. They cannot be calculated in the traditional way, for example, numerically using quadrature formulas. For example, the integrals with the kernels $U_{ij}(\mathbf{x}-\mathbf{y})$ are WS. They have to be considered as improper integrals. The integrals with the kernels $W_{ij}(\mathbf{x},\mathbf{y})$ and $K_{ij}(\mathbf{x},\mathbf{y})$ are singular. They have to be considered in the sense of Cauchy as PV. The integrals with the kernels $F_{ij}(\mathbf{x},\mathbf{y})$ are hypersingular. They have to be considered in the sense of Hadamard as FP. A traditional approach to the divergent integrals calculation may be found in [BaSlSl89, Ba94, MuMu05, TaSlSl94]. In [GuZo93, GuZo01, GuZo02, Zo91, Zo06a, Zo06b, Zo08, ZoGo99, ZoLu98, ZoMe00] we have developed the method of the divergent integral calculation based on theory of distribution. We will demonstrate here how this approach works in the problems of elastostatics.

31.4 Regularization of 1-D Divergent Integrals

This approach consists in application of the second Green theorem and transformation of divergent integrals into regular ones. In [Zo06a] we have developed formulas for regularization of the divergent integrals with singularities of the type r^{-m} in the form

$$F.P.\int_{-a}^{a}\frac{\varphi(x)}{r^m}dx$$
$$=\sum_{i=0}^{k-1}(-1)^{i+1}\frac{d^i}{dx^i}\frac{P_i}{r^{m-k}}\frac{d^{k-1-i}\varphi(x)}{dx^{k-1-i}}\Bigg|_{x=-a}^{x=a}+(-1)^k\int_{-a}^{a}\frac{P_k}{r^{m-k}}\frac{d^k\varphi(x)}{dx^k}. \quad (31.6)$$

In 2-D elastostatics, after introducing a local system of coordinates and simplification, all divergent integrals can be presented in the form

$$J_0=\int_a^b\varphi(x)\ln\frac{1}{x}dx,\quad J_k=\int_a^b\frac{\varphi(x)}{x^k}dx,\quad k=1,2.$$

Here $\varphi(x)$ is a smooth function that depends on the shape of the boundary element and interpolation polynomials.

We consider first the WS integral J_0. Because of logarithmical singularity, we cannot use formula (31.6). Therefore, we start from the formula for integration by parts in the form

$$\int_a^b \frac{dg(x)}{dx}\varphi(x)dx = \varphi(x)g(x)|_a^b - \int_a^b \frac{d\varphi(x)}{dx}g(x)dx. \tag{31.7}$$

In this formula, let $g(x) = x + x\ln\frac{1}{x}$, $\frac{dg(x)}{dx} = \ln\frac{1}{x}$, then we obtain

$$J_0 = W.S.\int_a^b \varphi(x)\ln\frac{1}{x}dx = \varphi(x)\left(x + x\ln\frac{1}{x}\right)\Big|_a^b - \int_a^b \frac{d\varphi(x)}{dx}\left(x + x\ln\frac{1}{x}\right)dx.$$

Obviously the integral on the left is divergent and the one on the right is regular. For linear boundary elements and a piecewise constant approximation, $\varphi(x) = 1$, and we get

$$J_0 = W.S.\int_a^b \ln\left|\frac{1}{x-y}\right|dx = (b-a) + (b-y)\ln\left|\frac{1}{b-y}\right| - (a-y)\ln\left|\frac{1}{a-y}\right|,$$

where $a < y < b$.

For the singular integral J_1, to achieve regularization we will also use the formula for integration by parts (31.7). Let $g(x) = -\ln\frac{1}{x}$, $\frac{dg(x)}{dx} = \frac{1}{x}$ in this formula; then we obtain

$$J_1 = P.V.\int_a^b \frac{\varphi(x)}{x}dx = \left(\frac{d\varphi(x)}{dx}x\ln\frac{1}{x} - \varphi(x)\ln\frac{1}{x}\right)\Big|_a^b - \int_a^b \frac{d^2\varphi(x)}{dx^2}x\ln\frac{1}{x}dx.$$

Here, the integral on the left is also divergent and that on the right is regular. For linear boundary elements and a piecewise constant approximation, $\varphi(x) = 1$, and we get

$$J_1 = P.V.\int_a^b \frac{dx}{x-y} = \ln\left|\frac{b-y}{a-y}\right|, \quad a < y < b.$$

Finally, for the hypersingular integral J_2, regularization is achieved by means of formula (31.6) and the above result for regularization of J_1. Finally, we get

$$J_2 = F.P.\int_a^b \frac{\varphi(x)}{x^2}dx$$
$$= \left(\frac{d^2\varphi(x)}{dx^2}x\ln\frac{1}{x} - \frac{d\varphi(x)}{dx}\ln\frac{1}{x} - \frac{\varphi(x)}{x}\right)\Big|_a^b - \int_a^b \frac{d^3\varphi(x)}{dx^3}x\ln\frac{1}{x}dx.$$

Here, the integral on the left is divergent and that on the right is regular. For linear boundary elements and a piecewise constant approximation, $\varphi(x) = 1$, and we get

$$J_2 = F.P. \int_a^b \frac{dx}{(x-y)^2} = -\frac{1}{b-y} + \frac{1}{a-y}, \quad a < y < b.$$

From this equation we can retrieve Hadamard's example of a function that is positive everywhere in the integration region but whose integral is negative:

$$F.P. \int_{-a}^{a} \frac{dy}{y^2} = -\frac{2}{a}, \quad a > 0.$$

31.5 Regularization of 2-D Divergent Integrals

This approach consists in application of the second Green theorem and transformation of divergent integrals into regular ones. In [Zo06a], we developed formulas for regularization of divergent integrals with singularities of the type r^{-m} in the form

$$F.P. \int_V \frac{\varphi(\mathbf{x})}{r^m} dV = \sum_{i=0}^{k-1} (-1)^{i+1} \int_{\partial V} \left[\Delta^{k-i-1} \varphi(\mathbf{x}) \partial_n \frac{P_i}{r^{m-2i}} - \frac{P_i}{r^{m-2i}} \partial_n \Delta^{k-i-1} \varphi(\mathbf{x}) \right] dS + (-1)^k \int_V \frac{1}{r^{m-2k}} \Delta^{k+1} \varphi(\mathbf{x})] dV. \quad (31.8)$$

In 3-D elastostatics, after introducing a local system of coordinates and simplification, all divergent integrals can be presented in the form

$$J_k^{l,m} = \int_{S_n} \frac{x_1^l x_2^m}{r^k} \varphi(x) dS, \quad l, m = 0,\ 1,\ 2, \quad k = 3,\ 4,\ 5.$$

Here, $\varphi(x)$ is a smooth function that depends on the shape of the boundary elements and interpolation polynomials.

31.5.1 Integrals with Kernels of the Type r^{-k}, $k = 1, 2, 3$

From equation (31.8) with $k = 1$ we get the regularization for the WS integral

$$J_1^{0,0} = W.S. \int_V \frac{\varphi(\mathbf{x})}{r} dV = \int_{\partial V} \left[\varphi(\mathbf{x}) \frac{r_n}{2r} - r \partial_n \varphi(\mathbf{x}) \right] dS + \int_V r \Delta \varphi(\mathbf{x}) dV. \quad (31.9)$$

Here $r_n = (x_\alpha - y_\alpha)n_\alpha$ and the summation convention applies to repeated indices $\alpha = 1, 2$. The integral on the left is divergent and the ones on the right are regular.

For the piecewise constant approximation, $\varphi(x) = 1$, and circular area, we can calculate this integral analytically. Introducing polar coordinates, we will get

$$J_1^{0,0} = \frac{1}{2} \int\limits_{\partial S_n} \frac{r_n}{r} dl = \int\limits_0^{2\pi} \frac{r}{r} r d\varphi = \pi r.$$

In order to regularize the singular integral, we will use the relation $\dfrac{1}{r^2} = \dfrac{1}{2}\Delta (\ln r)^2$. Then in the same way we get

$$\int\limits_V \frac{\varphi(\mathbf{x})}{r^2} dV = \frac{1}{2} \int\limits_{\partial V} \left(\varphi(\mathbf{x}) \frac{2r_n \ln r}{r^2} - (\ln r)^2 \partial_n \varphi(\mathbf{x}) \right) dS + \frac{1}{2} \int\limits_V (\ln r)^2 \Delta\varphi(\mathbf{x}) dV. \tag{31.10}$$

The volume integral on the right is WS. Taking into account relation $(\ln r)^2 = \dfrac{r^2}{6}\Delta (\ln r)^4$, we obtain regularization for this WS integral

$$\int\limits_V (\ln r)^2 \Delta\varphi(\mathbf{x}) dV = \frac{1}{6} \int\limits_{\partial V} \left(2\Delta\varphi(\mathbf{x}) r_n (\ln r)^3 - r^2 (\ln r)^4 \partial_n \Delta\varphi(\mathbf{x}) \right) dS + \frac{1}{6} \int\limits_V r^2 (\ln r)^4 \Delta^2 \varphi(\mathbf{x}) dV.$$

For a piecewise constant approximation, $\varphi(x) = 1$, and circular area, we can calculate the singular integral in (31.10) analytically. Introducing polar coordinates, we will get

$$J_2^{0,0} = \int\limits_{\partial S_n} \frac{r_n \ln r}{r^2} dl = \int\limits_0^{2\pi} \frac{r \ln r}{r^2} r d\varphi = 2\pi \ln r.$$

Finally, from equation (31.8) with $k = 3$, we get the regularization for the hypersingular integral

$$\int\limits_V \frac{\varphi(\mathbf{x})}{r^3} dV = \int\limits_{\partial V} \left[\Delta\varphi(\mathbf{x}) \frac{r_n}{2r} - \varphi(\mathbf{x}) \frac{r_n}{r^3} - \frac{1}{r} \partial_n \varphi(\mathbf{x}) - r \partial_n \Delta\varphi(\mathbf{x}) \right] dS + \int\limits_V r \Delta^2 \varphi(\mathbf{x}) dV. \tag{31.11}$$

For a piecewise constant approximation, $\varphi(x) = 1$, and circular area, we can calculate this integral analytically. Introducing polar coordinates, we will get

$$J_3^{0,0} = -\int_{\partial S_n} \frac{r_n}{r^3} dl = -\int_0^{2\pi} \frac{r}{r^3} r d\varphi = -\frac{2\pi}{r}.$$

Now using equation (31.8), any divergent integral with kernels of the type $1/r^k$for any positive integer k can be calculated.

31.5.2 Integrals with Kernels of the Type $\frac{x_\alpha^2}{r^k}$, $k = 3, 4, 5$

The WS integral with kernel $\frac{x_\alpha^2}{r^3}$ is calculated taking into account the equation $\frac{x_\alpha^2}{r^3} = \frac{1}{3}\left(\frac{2}{r} - \Delta\frac{x_\alpha^2}{r}\right)$. It is easy to show that

$$J_3^{2,0} = W.S.\int_V \varphi(\mathbf{x})\frac{x_1^2}{r^3} dV = \frac{2}{3} W.S.\int_V \frac{\varphi(\mathbf{x})}{r} dV - \frac{1}{3} W.S.\int_V \varphi(\mathbf{x})\Delta\frac{x_1^2}{r} dV. \tag{31.12}$$

The first integral here is already calculated in (31.9). The second one may be presented in the form

$$\int_V \varphi(\mathbf{x})\Delta\frac{x_1^2}{r} dV$$
$$= \int_{\partial V}\left[\varphi(\mathbf{x})\left(\frac{2n_1 x_1}{r} - \frac{x_1^2 r_n}{r^3}\right) - \frac{x_1^2}{r}\partial_n\varphi(\mathbf{x})\right] dS + \int_V \frac{x_1^2}{r}\Delta\varphi(\mathbf{x}) dV.$$

Combining the last two equations, we finally get

$$J_3^{2,0} = \frac{1}{3}\int_{\partial V}\left[\varphi(\mathbf{x})\left(\frac{2n_1 x_1}{r} - \frac{x_1^2 r_n}{r^3} + \frac{r_n}{2r}\right) - \left(\frac{x_1^2}{r} + r\right)\partial_n\varphi(\mathbf{x})\right] dS$$
$$+ \frac{1}{3}\int_V \left(\frac{x_1^2}{r} + r\right)\Delta\varphi(\mathbf{x}) dV.$$

For the piecewise constant approximation, $\varphi(x) = 1$, and in a circular area, we can calculate this integral analytically. Introducing polar coordinates, we get

$$J_3^{2,0} = \frac{1}{3}\int_{\partial S_n}\left(\frac{x_1^2 r_n}{r^3} - \frac{2x_1 n_1}{r} + \frac{2r_n}{r}\right) dl$$
$$= \frac{1}{3}\left(\int_0^{2\pi} \frac{(r\cos\varphi)^2 r}{r^3} r d\varphi - \int_0^{2\pi} \frac{2r(\cos\varphi)^2}{r} r d\varphi + 2\int_0^{2\pi} \frac{r}{r} r d\varphi\right) = \pi r.$$

The singular integral with kernel $\dfrac{x_\alpha^2}{r^4}$ is calculated taking into account that $\varDelta\dfrac{x_\alpha^2}{r^4} = \dfrac{1}{4}\left(\dfrac{2}{r^2} - \varDelta\dfrac{x_\alpha^2}{r^2}\right)$. In this case,

$$J_4^{2,0} = P.V.\int_V \varphi(\mathbf{x})\frac{x_1^2}{r^2}dV = \frac{1}{4}\int_V \varphi(\mathbf{x})\left(\frac{2}{r^2} - \varDelta\frac{x_1^2}{r^2}\right)dV$$
$$= \frac{1}{2}\int_V \frac{\varphi(\mathbf{x})}{r^2}dV - \frac{1}{4}\int_V \varphi(\mathbf{x})\varDelta\frac{x_1^2}{r^2}dV.$$

The first integral here is already calculated in (31.10). The second one may be presented in the form

$$\int_V \varphi(\mathbf{x})\varDelta\frac{x_1^2}{r^2}dV$$
$$= \int_{\partial V}\left[\varphi(\mathbf{x})\left(\frac{2n_1x_1}{r^2} - \frac{2x_1^2r_n}{r^4}\right) - \frac{x_1^2}{r^2}\partial_n\varphi(\mathbf{x})\right]dS + \int_V \frac{x_1^2}{r^2}\varDelta\varphi(\mathbf{x})dV.$$

Combining the last two equations, finally we will get

$$J_4^{2,0} = \frac{1}{2}\int_{\partial V}\left[\varphi(\mathbf{x})\left(\frac{x_1^2r_n}{r^4} - \frac{n_1x_1}{r^2} + \frac{r_n\ln r}{r^2}\right)\right.$$
$$\left. - \left(\frac{(\ln r)^2}{2} - \frac{x_1^2}{2r^2}\right)\partial_n\varphi(\mathbf{x})\right]dS + \frac{1}{4}\int_V\left((\ln r)^2 - \frac{x_1^2}{r^2}\right)\varDelta\varphi(\mathbf{x})dV.$$

For the piecewise constant approximation, $\varphi(x) = 1$, and circular area, we can calculate this integral analytically. Introducing polar coordinates, we will get

$$J_4^{2,0} = \frac{1}{2}\int_{\partial S_n}\left[\frac{x_1^2r_n}{r^4} - \frac{x_1n_1}{r^2} + \frac{r_n\ln r}{r^2}\right]dl$$
$$= \int_0^{2\pi}\frac{(r\cos\varphi)^2 r}{r^4}rd\varphi - \int_0^{2\pi}\frac{r(\cos\varphi)^2}{r^2}rd\varphi + \int_0^{2\pi}\frac{r\ln r}{r^2}rd\varphi = 2\pi\ln r.$$

The hypersingular integral with kernel $\dfrac{x_\alpha^2}{r^5}$ is calculated taking into account that $\dfrac{x_\alpha^2}{r^5} = \dfrac{1}{3}\left(\dfrac{2}{r^3} - \varDelta\dfrac{x_\alpha^2}{r^3}\right)$. In this case, it is easy to show that

$$J_5^{2,0} = F.P.\int_V \varphi(\mathbf{x})\frac{x_1^2}{r^5}dV = \frac{1}{3}\int_V \varphi(\mathbf{x})\left(\frac{2}{r^3} - \Delta\frac{x_1^2}{r^3}\right)dV$$
$$= \frac{2}{3}\int_V \frac{\varphi(\mathbf{x})}{r^3}dV - \frac{1}{3}\int_V \varphi(\mathbf{x})\Delta\frac{x_1^2}{r^3}dV.$$

The first integral here is already calculated in (31.11). The second one may be presented in the form

$$\int_V \varphi(\mathbf{x})\Delta\frac{x_1^2}{r^3}dV$$
$$= \int_{\partial V}[\varphi(\mathbf{x})\left(\frac{2n_1x_1}{r^3} - \frac{3x_1^2r_n}{r^5}\right) - \frac{x_1^2}{r^3}\partial_n\varphi(\mathbf{x})]dS + \int_V \frac{x_1^2}{r^3}\Delta\varphi(\mathbf{x})dV.$$

Combining the last two equations, we finally get

$$J_5^{2,0} = \frac{2}{3}\int_{\partial V}\left(\varphi(\mathbf{x})\left(\frac{n_1x_1}{r^3} - \frac{r_n}{r^3} - \frac{3x_1^2r_n}{2r^5}\right) - \left(\frac{1}{r} + \frac{x_1^2}{r^3}\right)\partial_n\varphi(\mathbf{x})\right)dS$$
$$+ \frac{2}{3}W.S.\int_V \left(\frac{1}{r} - \frac{x_1^2}{r^3}\right)\Delta\varphi(\mathbf{x})dV.$$

The volume integral here is WS. Its regularization may be achieved using the equations (31.9) and (31.12). For the linear boundary element and piecewise constant approximation, $\varphi(x) = 1$, and circular area, we can calculate this integral analytically. Introducing polar coordinates, we get

$$J_5^{2,0} = \int_{\partial S_n}\left(\frac{2r_n}{3r^3} + \frac{2x_1^2r_n}{3r^5} - \frac{x_1n_1}{r^3}\right)dl$$
$$= \frac{2}{3}\int_0^{2\pi}\frac{r}{r^3}rd\varphi + \frac{2}{3}\int_0^{2\pi}\frac{(r\cos\varphi)^2 r}{r^5}rd\varphi - \int_0^{2\pi}\frac{r(\cos\varphi)^2}{r^3}rd\varphi = -\frac{\pi}{r}.$$

31.5.3 Integrals with Kernels of the Type $\frac{x_1x_2}{r^k}$, $k = 3, 4, 5$

The WS integral with kernel $\frac{x_1x_2}{r^3}$ is calculated using equation (31.8). Taking into account that $\frac{x_1x_2}{r^3} = -\frac{1}{3}\Delta\frac{x_1x_2}{r}$, it is easy to show that

$$J_3^{1,1} = W.S.\int_V \varphi(\mathbf{x})\frac{x_1x_2}{r^3}dV$$
$$= \frac{1}{3}\int_{\partial V}\left[\varphi(\mathbf{x})\left(\frac{x_1x_2r_n}{r^3} - \frac{r_*}{r}\right) + \frac{x_1x_2}{r}\partial_n\varphi(\mathbf{x})\right]dS - \frac{1}{3}\int_V \frac{x_1x_2}{r}\Delta\varphi(\mathbf{x})dV.$$

Here $r_* = x_1 n_2 + x_2 n_1$.

For a piecewise constant approximation, $\varphi(x) = 1$, and for a circular area, we can calculate this integral analytically. Introducing polar coordinates, we get

$$J_3^{1,1} = \frac{1}{3} \int\limits_{\partial S_n} \left[\frac{x_1 x_2 r_n}{r^3} - \frac{r_*}{r} \right] dl$$

$$= \int\limits_0^{2\pi} \frac{r^3 \cos\varphi \sin\varphi}{r^3} r d\varphi - \int\limits_0^{2\pi} \frac{2r \cos\varphi \sin\varphi}{r} r d\varphi = 0.$$

The singular integral with kernel $\dfrac{x_1 x_2}{r^4}$ is calculated using equation (31.8). Taking into account that $\dfrac{x_1 x_2}{r^4} = -\dfrac{1}{4} \Delta \dfrac{x_1 x_2}{r^2}$, it is easy to show that

$$\begin{aligned} J_4^{1,1} &= P.V. \int\limits_V \varphi(\mathbf{x}) \frac{x_1 x_2}{r^4} dV \\ &= \frac{1}{4} \int\limits_{\partial V} \left[\varphi(\mathbf{x}) \left(\frac{2 x_1 x_2 r_n}{r^4} - \frac{r_*}{r^2} \right) - \frac{x_1 x_2}{r^2} \partial_n \varphi(\mathbf{x}) \right] dS \\ &\quad - \frac{1}{4} \int\limits_V \frac{x_1 x_2}{r^2} \Delta \varphi(\mathbf{x}) dV. \end{aligned} \tag{31.13}$$

For a piecewise constant approximation, $\varphi(x) = 1$, and, if the area is circular, we can calculate this integral analytically. Introducing polar coordinates, we get

$$\begin{aligned} J_4^{1,1} &= \frac{1}{4} \int\limits_{\partial S_n} \left[\frac{2 x_1 x_2 r_n}{r^4} - \frac{r_*}{r^2} \right] dl \\ &= \frac{1}{4} \int\limits_0^{2\pi} \frac{2r^3 \cos\varphi \sin\varphi}{r^4} r d\varphi - \frac{1}{4} \int\limits_0^{2\pi} \frac{2r \cos\varphi \sin\varphi}{r^2} r d\varphi = 0. \end{aligned}$$

The hypersingular integral with kernel $\dfrac{x_1 x_2}{r^5}$ is calculated using equation (31.8). Taking into account that $\dfrac{x_1 x_2}{r^5} = -\dfrac{1}{3} \Delta \dfrac{x_1 x_2}{r^3}$, it is easy to show that

$$
\begin{aligned}
J_5^{1,1} &= F.P. \int\limits_V \varphi(\mathbf{x}) \frac{x_1 x_2}{r^5} dV \\
&= \int\limits_{\partial V} \left[\varphi(\mathbf{x}) \left(\frac{x_1 x_2 r_n}{r^5} - \frac{r_*}{3r^3} \right) - \frac{x_1 x_2}{r^3} \partial_n \varphi(\mathbf{x}) \right] dS \\
&\quad - \frac{1}{3} \int\limits_V \frac{x_1 x_2}{r^3} \Delta\varphi(\mathbf{x}) dV.
\end{aligned}
$$

The volume integral here is WS. Its regularization may be achieved using equation (31.13).

For a piecewise constant approximation, $\varphi(x) = 1$, and, if the area is circular, we can calculate this integral analytically. Introducing polar coordinates, we get

$$
J_5^{1,1} = \int\limits_{\partial S_n} \left[\frac{x_1 x_2 r_n}{r^5} - \frac{r_*}{3r^3} \right] dl
$$

$$
= \int\limits_0^{2\pi} \frac{r^3 \cos\varphi \sin\varphi}{r^5} r d\varphi - \int\limits_0^{2\pi} \frac{2r \cos\varphi \sin\varphi}{3r^3} r d\varphi = 0.
$$

In [Zo06b, Zo08, ZoGo99, ZoLu98, ZoMe00], it was shown that divergent integrals over any polygonal area may be calculated analytically in a similar way.

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